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AN

INTRODUCTORY TREATISE

TO

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AN

INTRODUCTORY TREATISE

TO

PHYSICAL ASTRONOMY.

BIBLIOTHECA BODLEIANA

By THOMAS LUBY, A. M.

TRINITY COLLEGE, DUBLIN.

> "*Prout Physica, majora indies incrementa capiet et nova axiomata educet; eo mathematicæ operá nová in multis indigebit, et plures demum fient mathematicæ mixtæ.*"
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PREFACE.

IN the courses of education pursued in the Universities of these countries, the study of Physical Astronomy holds a most prominent place. In conducting this study there are two methods pursued—either the geometrical, as in the Principia of Newton, or the analytical, as in publications of a more recent date. Though a knowledge of both these methods is required of students, no *single* work has, however, been published to enable them to acquire this knowledge with facility. The design of the author in venturing to offer the present work to the notice of the public has been to remedy this deficiency, to furnish an elementary treatise which might serve to initiate students in the principles of Physical Astronomy, which might set before them, in a connected state, the two modes of conducting its processes that have generally been used, and thus enable them to form a comparative estimate of their respective merits. A great deal, no doubt, may be found on this subject amongst the papers that have been at different times published in the transactions of learned societies; a great deal, also, may be found introduced, in the way of exemplification, amongst the many works that have from time to time been published on the Calculus, on Mechanics, and on the more recondite parts of Physical Astronomy; but to derive advantage from such sources, there is required an expenditure of time and of labour which the generality of students are neither able nor willing to afford. There is, moreover, an irksomeness of a peculiar nature attendant on this labour, arising from the variety of method and of notation which different writers have thought proper to adopt. To the publication of such a work as this, the small share of preparatory knowledge possessed by the students on the subject of Mechanical Philosophy has hitherto operated as a powerful preventive. It would be difficult, as well as tedious, to pass from the Mechanics of Helsham or Wood to the discussions of Physical Astronomy. This difficulty has, however, been removed, at least in our University, by the excellent Treatise on Mechanics furnished by the present Professor of Natural Philosophy; a treatise, in which the deepest research is united with that clearness of exposition, which constitutes the chief ornament of a work intended for elementary instruction.

This work consists of two parts :—in part the first the author has given selections from Newton, as far as is comprised within the medal course in Dublin College; part the second, by much the more extensive of the two, is a new Analytical Treatise on the elementary parts of Physical Astronomy, in which it has been attempted to render this subject as full and as explicit as the nature of it would admit. There are notes to both these parts; and the ninth section of Newton is introduced as an appendix, in expectation that it will ere long constitute formally a part of the medal course.

The materials for the present work have been selected from every source within the author's reach; he begs leave, however, more particularly to acknowledge himself indebted to the papers of Dr. Brinkley, the present Bishop of Cloyne. He has also derived considerable assistance from the works of Simpson, Mac Laurin, Woodhouse, and others. It is for those who feel interested in the subject to decide how far he has succeeded in his attempt, and what are the merits, original or derived, which his work may possess. It has been, both previously to, and during the course of publication, applied to the purposes of education with success. The consideration of this circumstance, with the necessity of some such work at the present time, will, he trusts, rescue him from the imputation of presumption in the attempt.

DUBLIN COLLEGE,
May 1, 1828.

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PART I.

The Reader is requested to make the following Corrections.

Page 8, *l.* 2 from bottom, *for* B *read* D

9, *l.* last, *for* pN as *read* PN as

42, *l.* 9, *for* nV *read* uV

103, *l.* 3 from bottom, *for* $2ar+r^2$ *read* $ar+2r^2$

108, *l.* 7, *for* dx *read* $x\,dx$

—, *l.* 8, *for* dt *read* t

110, *l.* 6, *for* $(l-r'^n)$ *read* $(l-r')^n$

114, *l.* 11, *for* $\frac{x-a}{a}$ *read* $\frac{x-a}{r}$

121, *l.* last, *for* $\frac{h^2}{2}$ *read* h^2

126, *ll.* 16 and 21, *for* μ log. *read* 2μ. log.

—, *ll.* 18 and 23, *for* $r\,dp$ *read* $2\,r\,dp$

128, *l.* 9, *for* $\pm$ *read* $\mp$

132, *lll.* 8, 14, and 15, *for* 2 *read* $\frac{1}{2}$

134, *lll.* 9, 10, and 11, *for* $(\mathrm{P}.v)^2$ *read* $\frac{1}{2}.(\mathrm{P}.v)^2$

136, *l.* 19, *for* $(1-e^2)$ *read* $\sqrt{1-e^2}$

140, *l.* 7, *for* $=$ *read* $-$

148, *l.* 14, *for* $\frac{\cos.m\omega}{(1-e^2)}$ *read* $\frac{\cos.m\omega.d\omega}{(1-e^2)^2}$

149, *l.* 12, *for* $-$ *read* $+$

153, *l.* 10, *for* t *read* $\cos.\phi.t$

156, *ll.* 9 and 12, *for* $x\tan.\phi+b$ *read* $(x+b).\tan.\phi$

—, *l.* 24, supply $y=x.\tan.\phi+b$

162, *l.* 18, *for* $\pm$ *read* $\mp$

164, *ll.* 3 and 5, *for* b *read* $2b$

168, *ll.* 17 and 18, *for* velocity *read* (velocity)^2

177, *ll.* 2 and 3 from bottom, *for* $-2\pi, -4\pi$ *read* $-\pi, -2\pi$

178, *l.* 16, *for* k *read* $\frac{1}{k}$

179, *ll.* 14 and 17, *for* $\frac{1}{\sqrt{\mu}}$ *read* $\frac{l^2.\tan.^2\theta}{\sqrt{\mu}}$

—, *l.* 20, *for* k *read* $\frac{1}{k}$

183, *ll.* 16 and 19, *for* $\frac{1}{h\gamma}$ *read* $\frac{a}{h\gamma}$

184, *l.* 12, *for* γ^2 *read* $m^2\gamma^2$

—, *ll.* 13 and 15, *for* $\frac{1}{\gamma}$ *read* $\frac{1}{m\gamma}$

187, *ll.* 14 and 16, *for* $\frac{\sqrt{\mu}}{2\sqrt{2}.\rho}$ *read* $\frac{2\sqrt{2}.\rho}{\sqrt{\mu}}$

—, *l.* 19, *for* $\frac{\pi.\rho\sqrt{\mu}}{2\sqrt{2}}$ *read* $\frac{2\sqrt{2}.\rho^2.\pi}{\sqrt{\mu}}$

194, *ll.* 13 and 15, *for* n *read* $(n-1)$

196, *ll.* 21 and 22, *for* k *read* $\frac{k.h^4}{\mu}$

—, *l.* 25, *for* γ^2u^2 *read* $k^2\gamma^2u^2$

208, *l.* 20, *for* $\frac{\pi a^2}{2h}$ *read* $\frac{3\pi a^2}{4h}$

222, *l.* 22, *for* lines *read* line r

230, *ll.* 9 and 10, x^2 *read* c^2x^2

AN

INTRODUCTORY TREATISE

TO

PHYSICAL ASTRONOMY.

PART I.

SELECTIONS FROM NEWTON.

Definitions.

Def. I.—*The quantity of matter is the measure of the same, arising from its density and bulk conjointly.*

THUS air of a double density, in a double space, is quadruple in quantity; in a triple space, sextuple in quantity. The same thing is to be understood of snow, and fine dust or powders, that are condensed by compression or liquefaction; and of all bodies that are by any causes whatever differently condensed. I have no regard in this place to a medium, if any such there be, that freely pervades the interstices between the parts of bodies. It is this quantity that I mean hereafter every where under the name of Body or Mass. And the same is known by the weight of each body: for it is proportional to the weight, as I have found by experiments on pendulums, very accurately made, which shall be shewn hereafter.

II.—*The quantity of motion is the measure of the same, arising from the velocity and quantity of matter conjointly.*

The motion of the whole is the sum of the motions of all the parts; and therefore in a body double in quantity, with equal velocity, the motion is double; with twice the velocity, it is quadruple.

III.—*The* vis insita, *or innate force of matter, is a power of resisting, by which every body, as much as in it lies, endeavours to persevere in its present state, whether it be of rest, or of moving uniformly forward in a right line.*

This force is ever proportional to the body whose force it is; and differs nothing from the inactivity of the mass, but in our manner of conceiving it. A body from the inactivity of matter, is not without difficulty put out of its state of rest or motion. Upon which account, this *vis insita* may, by a most significant name, be called *vis inertiæ*, or force of inactivity. But a body exerts this force only when another force, impressed upon it, endeavours to change its condition; and the exercise of this force may be considered both as resistance and impulse: it is *resistance*, in so far as the body, for maintaining its present state, withstands the force impressed; it is *impulse*, in so far as the body, by not easily giving way to the impressed force of another, endeavours to change the state of that other. Resistance is usually ascribed to bodies at rest, and impulse to those in motion: but motion and rest, as commonly conceived, are only relatively distinguished; nor are those bodies always truly at rest, which commonly are taken to be so.

IV.—*An impressed force is an action exerted upon a body, in order to change its state, either of rest, or of moving uniformly forward in a right line.*

This force consists in the action only; and remains no longer in the body, when the action is over. For a body maintains every new state it acquires, by its *vis inertiæ* only. Impressed forces are of different origins; as from percussion, from pressure, from centripetal force.

V.—*A centripetal force is that by which bodies are drawn or impelled, or any way tend towards a point as to a centre.*

Of this sort is gravity, by which bodies tend to the centre of the earth; magnetism, by which iron tends to the load stone; and that force, whatever it be, by which the planets are perpetually drawn aside from the rectilinear motions, which otherwise they would pursue, and made to revolve in curvilinear orbits. A stone, whirled about in a sling, endeavours to recede from the hand that turns it; and by that endeavour, distends the sling, and that with so much the greater force, as it is revolved with the greater velocity, and as soon as it is let go flies away. That force which opposes itself to this endeavour, and by which the sling perpetually draws back the stone towards the hand, and retains it in its orbit, because it is directed to the hand as the centre of the orbit, I call the centripetal force. And the same thing is to be understood of all bodies, revolving in any orbits. They all endeavour to recede from the centres of their orbits; and were it not for the opposition of a contrary force which restrains them to, and detains them in their orbits, which I therefore call centripetal, would fly off in right lines, with an uniform motion. A projectile, if it were not for the force of gravity, would not deviate towards the earth, but would go off from it in a right line, and that with an uniform motion, if the resistance of the air were taken away. It is by its gravity that it is drawn aside perpetually from its rectilinear course, and made to deviate towards the earth, more or less, according to the force of its gravity, and the velocity of its motion. The less its gravity is, for the quantity of its matter, or the greater the velocity with which it is projected, the less will it deviate from a rectilinear course, and the farther it will go. If a leaden ball, projected from the top of a mountain by the force of gunpowder with a given velocity, and in a direction parallel to the horizon, be carried in a curve line to the distance of two miles before it falls to the ground; the same, if the resistance of the air were removed, with a double or decuple velocity, would fly twice or ten times as far. And by increasing the velocity, we may at pleasure increase the distance to which it would be projected, and diminish the curvature of the line, which it would describe, till at last it should fall at the distance of ten, thirty, or ninety degrees, or even might go quite round the whole earth; or lastly, go forward into the celestial spaces, and proceed in its motion *in infinitum.* And after the same manner that a projectile, by the force of gravity, may be made to revolve in an orbit, and go round the whole earth, the moon also, either by the force of gravity, if it be endued with gravity, or by any other force that impels it towards the earth, may be perpetually drawn aside towards the earth, out of the rectilinear course, which by its innate force it would pursue; and be made to revolve in the orbit which it now describes: nor could the moon without some such force be retained in its orbit. If this force were too small, it would not sufficiently turn the moon out of a rectilinear course: if it were too great, it would turn it too much, and draw down the moon from its orbit towards the earth. It is necessary that the force be of a just quantity, and it belongs to mathematicians to find the force that may serve exactly to retain a body in a given orbit, with a given velocity; and *vice versâ*, to determine the curvilinear path into which a body projected from a given place, with a given velocity, may be made to deviate from its natural rectilinear path, by means of a given force.

The quantity of any centripetal force may be considered as of three kinds; *absolute*, *accelerative*, and *motive.*

VI.—*The absolute quantity of a centripetal force is the measure of the same, proportional to the efficacy of the cause that propagates it from the centre, through the spaces round about.*

Thus the magnetic force is greater in one load-stone and less in another, according to their sizes and strength.

VII.—*The accelerative quantity of a centripetal force is the measure of the same, proportional to the velocity which it generates in a given time.*

Thus the force of the same load-stone is greater at a less distance, and less at a greater: also, the force of gravity is greater in valleys, less on tops of exceeding high mountains; and yet less (as shall be hereafter shewn) at greater distances from the body of the earth; but at equal distances it is the same every where; because (taking away, or allowing for, the resistance of the air) it equally accelerates all falling bodies, whether heavy or light, great or small.

VIII.—*The motive quantity of a centripetal force, is the measure of the same, proportional to the motion which it generates in a given time.*

Thus the weight is greater in a greater body, less in a less body; it is greater near to the earth, and less at remoter distances. This sort of quantity is the centripetency, or propension of the whole body towards the centre, or, as I may say, its weight; and it is ever known by the quantity of a force equal and contrary to it, that is just sufficient to hinder the descent of the body.

These quantities of forces we may, for brevity sake, call by the names of *motive*, *accelerative*, and *absolute* forces; and, for distinction sake, consider them, with respect to the bodies that tend to the centre; to the places of those bodies; and to the centre of force towards which they tend: that is to say, I refer the *motive* force to the body, as an endeavour and propensity of the whole towards a centre, arising from the propensities of the several parts taken together; the *accelerative* force to the place of the body, as a certain power or energy diffused from the centre to all places around to move the bodies that are in them; and the *absolute* force to the centre, as endued with some cause, without which those motive forces would not be propagated through the spaces round about; whether that cause is some central body (such as is the load-stone, in the centre of the force of magnetism, or the earth in the centre of the gravitating force) or any thing else that does not yet appear. For I here design only to give a mathematical notion of those forces, without considering their physical causes and seats.

Wherefore, the accelerative force will stand in the same relation to the motive, as velocity does to motion. For the quantity of motion arises from the velocity into the quantity of matter; and the motive force arises from the accelerative force into the same quantity of matter. For the sum of the actions of the accelerative force, upon the several particles of the body, is the motive force of the whole. Hence it is, that near the surface of the earth, where the accelerative gravity, or force productive of gravity, in all bodies is the same, the motive gravity or the weight is as the body: but if we should ascend to higher regions, where the accelerative gravity is less, the weight would be likewise diminished, and would always be as the product of the body, by the accelerative gravity. So in those regions, where the accelerative gravity is di-diminished into one-half, the weight of a body two or three times less, will be four or six times less.

I likewise call attractions and impulses, in the same sense, accelerative and motive; and use the words attraction, impulse, or propensity of any sort towards a centre, promiscuously and indifferently, one for another; considering those forces not physically, but mathematically: wherefore, the reader is not to imagine that, by those words, I any where take upon me to define the kind, or the manner of any action, the causes or the physical reason thereof, or that I attribute forces, in a true and physical sense, to certain centres, (which are only mathematical points); when at any time I happen to speak of centres as attracting, or as endued with attractive powers.

AXIOMS; OR LAWS OF MOTION.

LAW I.

Every body perseveres in its state of rest, or of uniform motion in a right line, unless it is compelled to change that state by forces impressed thereon.

LAW II.

The alteration of motion is ever proportional to the motive force impressed; and is made in the direction of the right line in which that force is impressed.

LAW III.

To every action there is always opposed an equal reaction; or the mutual actions of two bodies upon each other are always equal, and directed to contrary parts.

COR. I.—*A body by two forces conjoined will describe the diagonal of a parallelogram, in the same time that it would describe the sides, by those forces apart.* (Fig. 1.)

If a body in a given time, by the force M impressed apart in the place A, should with an uniform motion be carried from A to B; and by the force N impressed apart in the same place, should be carried from A to C; complete the parallelogram ABCD, and, by both forces acting together, it will in the same time be carried in the diagonal from A to D. For since the force N acts in the direction of the line AC, parallel to BD, this force (by the second law) will not at all alter the velocity generated by the other force M, by which the body is carried towards the line BD. The body, therefore, will arrive at the line BD in the same time, whether the force N be impressed or not; and therefore at the end of that time it will be found somewhere in the line BD. By the same argument, at the end of the same time it will be found somewhere in the line CD. Therefore it will be found in the point D, where both lines meet. But it will move in a right line from A to D, by Law I.

II.—*And hence is explained the composition of any one direct force* AD, *out of any two oblique forces* AB *and* BD; *and, on the contrary, the resolution of any one direct force* AD *into two oblique forces* AB *and* BD: *which composition and resolution are abundantly confirmed from mechanics.* (Fig. 2.)

As if the unequal radii OM and ON drawn from the centre O of any wheel, should sustain the weights A and P by the cords MA and NP; and the forces of those weights to move the wheel were required. Through the centre O draw the right line KOL, meeting the cords perpendicularly in K and L; and from the centre O, with OL the greater of the distances OK and OL, describe a circle, meeting the cord MA in D: and drawing OD, make AC parallel and DC perpendicular thereto. Now, it being indifferent whether the points K, L, B, of the cords be fixed to the plane of the wheel or not, the weights will have the same effect whether they are suspended from the points K and L, or from D and L. Let the whole force of the weight A be represented by the line AD, and let it be resolved into the forces AC and CD; of which the force AC, drawing the radius OD directly from the centre, will have no effect to move the wheel: but the other force DC, drawing the radius DO perpendicularly, will have the same effect as if it drew perpendicularly the radius OL equal to OD; that is, it will have the same effect as the weight P, if that weight be to the weight A as the force DC is to the force DA; that is, (because of the similar triangles ADC, DOK,) as OK to OD or OL. Therefore, the weights A and P, which are reciprocally as the radii OK and OL that lie in the same right line, will be equipollent, and so remain in equilibrio: which is the well known property of the balance, the lever, and the wheel. If either weight be greater than in this ratio, its force to move the wheel will be so much the greater.

If the weight p, equal to the weight P, be partly suspended by the cord Np, partly sustained by the oblique plane pG; draw pH, NH, the former perpendicular to the horizon, the latter to the plane pG; and if the force of the weight p tending downwards be represented by the line pH, it may be resolved into the forces pN, HN. If there were any plane pQ perpendicular to the cord pN, cutting the other plane pG in a line parallel to the horizon, and the weight p was supported only by those planes pQ, pG, it would press those planes perpendicularly with the forces pN, HN; to wit, the plane pQ with the force pN, and the plane pG with the force HN. And therefore if the plane pQ were taken away, so that the weight might stretch the cord, because the cord, now sustaining the weight, supplies the place of the plane that was removed, it will be strained by the same force pN which pressed upon the plane before. Therefore the tension of this oblique cord pN will be to that of the other perpendicular cord pN as pN to pH. And therefore if the weight p is to the weight A in a ratio compounded of the reciprocal ratio of the least distances of the cords pN, AM, from the centre of the wheel, and of the direct ratio of pH to pN, the weights will have the same effect towards moving the wheel, and will therefore sustain each other; as any one may find by experiment.

But the weight p pressing upon those two oblique planes, may be considered as a wedge between the two internal surfaces of a body split by it; and hence the forces of the wedge and the mallet may be determined; for because the force with which the weight p presses the plane pQ is to the force with which the same, whether by its own gravity, or by the blow of a mallet, is impelled in the direction of the line pH towards both the planes, as pN to pH; and to the force with which it presses the other plane pG, as pN to NH. And thus the force of the screw may be deduced from a like resolution of forces; it being no other than a wedge impelled with the force of a lever. Therefore the use of this Corollary spreads far and wide, and by that diffusive extent the truth thereof is farther confirmed. For on what has been said depends the whole doctrine of mechanics variously demonstrated by different authors. For from hence are easily deduced the forces of machines, which are compounded of wheels, pullies, levers, cords, and weights, ascending directly or obliquely, and other mechanical powers; as also the force of the tendons to move the bones of animals.

III.—*The quantity of motion, which is collected by taking the sum of the motions directed towards the same parts, and the difference of those that are directed to contrary parts, suffers no change from the action of bodies among themselves.*

For action and its opposite reaction are equal, by Law 3, and therefore, by Law 2, they produce in the motions equal changes towards opposite parts. Therefore if the motions are directed towards the same parts, whatever is added to the motion of the preceding body will be subducted from the motion of that which follows; so that the sum will be the same as before. If the bodies meet, with contrary motions, there will be an equal deduction from the motions of both; and therefore the difference of the motions directed towards opposite parts will remain the same.

IV.—*The common centre of gravity of two or more bodies does not alter its state of motion or rest by the actions of the bodies among themselves; and therefore the common centre of gravity of all bodies acting upon each other (excluding outward actions and impediments) is either at rest, or moves uniformly in a right line.*

For if two points proceed with an uniform motion in right lines, and their distance be divided in a given ratio, the dividing point will be either at rest, or proceed uniformly in a right line. This is demonstrated hereafter in Lem. 23 and its Corol., when the points are moved in the same plane; and by a like way of arguing, it may be demonstrated when the points are not moved in the same plane. Therefore if any number of bodies move uniformly in right lines, the common centre of gravity of any two of them is either at rest, or proceeds uniformly in a right line; because the line which connects the centres of those two bodies so moving is divided at that common centre in a given ratio. In like manner the common centre of those two and that of a third body will be either at rest or moving uniformly in a right line; because at that centre the distance between the common centre of the two bodies, and the centre of this last, is divided in a given ratio. In like manner the common centre of these three, and of a fourth body, is either at rest, or moves uniformly in a right line; because the distance between the common centre of the three bodies, and the centre of the fourth is there also divided in a given ratio, and so on *in infinitum*. Therefore in a system of bodies where there is neither any mutual action among themselves, nor any foreign force impressed upon them from without, and which consequently move uniformly in right lines, the common centre of gravity of them all is either at rest, or moves uniformly forwards in a right line.

Moreover, in a system of two bodies mutually acting upon each other, since the distances between their centres and the common centre of gravity of both are reciprocally as the bodies, the relative motions of those bodies, whether of approaching to, or of receding from, that centre, will be equal among themselves. Therefore since the changes which happen to motions are equal and directed to contrary parts, the common centre of those bodies, by their mutual action between themselves, is neither promoted nor retarded, nor suffers any change as to its state of motion or rest. But in a system of several bodies, because the common centre of gravity of any two acting mutually upon each other suffers no change in its state by that action; and much less the common centre of gravity of the others with which that action does not intervene; but the distance between those two centres is divided by the common centre of gravity of all the bodies into parts reciprocally proportional to the total sums of those bodies whose centres they are; and therefore while those two centres retain their state of motion or rest, the common centre of all does also retain its state: it is manifest that the common centre of all never suffers any change in the state of its motion or rest from the actions of any two bodies between themselves. But in such a system all the actions of the bodies among themselves either happen between two bodies, or are composed of actions interchanged between some two bodies; and therefore they do never produce any alteration in the common centre of all as to its state of motion or rest. Wherefore since that centre, when the bodies do not act mutually one upon another, either is at rest or moves uniformly forward in some right line, it will, notwithstanding the mutual actions of the bodies among themselves, always persevere in its state, either of rest, or of proceeding uniformly in a right line, unless it is forced out of this state by the action of some power impressed from without upon the whole system. And therefore the same law takes place in a system consisting of many bodies as in one single body, with regard to their persevering in their state of motion or of rest. For the progressive motion, whether of one single body, or of a whole system of bodies, is always to be estimated from the motion of the centre of gravity.

V.—*The motions of bodies included in a given space are the same among themselves, whether that space is at rest, or moves uniformly forwards in a right line without any circular motion.*

For the differences of the motions tending towards the same parts, and the sums of those that tend towards contrary parts are, at first (by supposition) in both cases the same; and it is from those sums and differences that the collisions and impulses do arise with which the bodies mutually impinge one upon another. Wherefore (by Law 2) the effects of those collisions will be equal in both cases; and therefore the mutual motions of the bodies among themselves in the one case will remain equal to the mutual motions of the bodies among themselves in the other. A clear proof of which we have from the experiment of a ship; where all motions happen after the same manner, whether the ship is at rest, or is carried uniformly forwards in a right line.

VI.—*If bodies, any how moved among themselves, are urged in the direction [illegible] parallel lines by equal accelerative* *forces, they will all continue to move among themselves, after the same manner as if they had been urged by no such forces.*

For these forces acting equally, (with respect to the quantities of the bodies to be moved,) and in the direction of parallel lines, will (by Law 2) move all the bodies equally, (as to velocity,) and therefore will never produce any change in the positions or motions of the bodies among themselves.

OF THE MOTION OF BODIES.

BOOK I.

SECTION I.

Of the method of prime and ultimate ratios of quantities, by the help whereof we demonstrate the propositions that follow.

LEMMA I.

Quantities, and the ratios of quantities, which in any finite time converge continually to equality, and before the end of that time approach nearer the one to the other than by any given difference, become ultimately equal.

If you deny it, suppose them to be ultimately unequal, and let D be their ultimate difference. Therefore they connot approach nearer to equality than by that given difference D; which is against the supposition.

LEMMA II.

If in any figure AacE (Fig. 3.) *terminated by the right lines* Aa, AE, *and the curve* acE, *there be inscribed any number of parallelograms* Ab, Bc, Cd, *&c. comprehended under equal bases* AB, BC, CD, *&c. and the sides* Bb, Cc, Dd, *&c. parallel to one side* Aa *of the figure; and the parallelograms* aKbl, bLcm, cMdm, *&c. be completed. Then if the breadth of those parallelograms be supposed to be diminished, and their number to be augmented* in infinitum; *I say, that the ultimate ratios which the inscribed figure* AKbLcMdD, *the circumscribed figure* AalbmcndoE, *and curvilinear figure* AabcdE, *will have to one another, are ratios of equality.*

For the difference of the inscribed and circumscribed figures is the sum of the parallelograms Kl, Lm, Mn, Do, that is, (from the equality of all their bases,) the rectangle under one of their bases Kb and the sum of their altitudes Aa, that is, the rectangle ABla. But this rectangle, because its breadth AB is supposed diminished *in infinitum*, becomes less than any given space. And therefore (by Lem. 1.) the figures inscribed and circumscribed become ultimately equal one to the other; and much more will the intermediate curvilinear figure be ultimately equal to either. Q. E. D.

LEMMA III.

The same ultimate ratios are also ratios of equality, when the breadths AB, BC, DC, *&c. of the parellelograms are unequal, and are all diminished* in infinitum.

For suppose AF equal to the greatest breadth, and complete the parallelogram FAaf. This parallelogram will be greater than the difference of the inscribed and circumscribed figures; but because its breadth AF is diminished *in infinitum*, it will become less than any given rectangle. Q. E. D.

COR. 1. Hence the ultimate sum of those evanescent parallelograms will in all parts coincide with the curvilinear figure.

COR. 2. Much more will the rectilinear figure comprehended under the chords of the evanescent arcs ab, bc, cd, &c. ultimately coincide with the curvilinear figure.

COR. 3. And also the circumscribed rectilinear figure comprehended under the tangents of the same arcs.

COR. 4. And therefore these ultimate figures (as to their perimeters acE) are not rectilinear, but curvilinear limits of rectilinear figures.

LEMMA IV.

If in two figures AacE, PprT, (Fig. 4.) *you inscribe, as before, two ranks of parallelograms, an equal number in each rank, and, when their breadths are diminished* in infinitum, *the ultimate ratios of the parallelograms in one figure to those in the other, each to each respectively, are the same; I say, that those two figures* AacE, PprT, *are to one another in that same ratio.*

For as the parallelograms in the one are severally to the parallelograms in the other, so (by composition) is the sum of all in the one to the sum of all in the other; and so is the one figure to the other; because (by Lem. 3.) the former figure to the former sum, and the latter figure to the latter sum, are both in the ratio of equality. Q. E. D.

COR. Hence if two quantities of any kind are any how divided into an equal number of parts, and those parts, when their number is augmented, and their magnitude diminished *in infinitum*, have a given ratio one to the other, the first to the first, the second to the second, and so on in order, the whole quantities will be one to the other in that same given ratio. For if, in the figures of this lemma, the parallelograms are taken one to the other in the ratio of the parts, the sum of the parts will always be as the sum of the parallelograms; and therefore supposing the number of the parallelograms and parts to be augmented, and their magnitudes diminished *in infinitum*, those sums will be in the ultimate ratio of the parallelogram in the one figure to the corresponding parallelogram in the other; that is, (by the supposition,) in the ultimate ratio of any part of the one quantity to the corresponding part of the other.

LEMMA V.

In similar figures, all sorts of homologous sides, whether curvilinear or rectilinear, are proportional; and the areas are in the duplicate ratio of the homologous sides.

LEMMA VI.

If any arc ACB (Fig. 5.) *given in position be subtended by its chord* AB, *and in any point* A, *in the middle of the continued curvature, be touched by a right line* AD, *produced both ways; then if the points* A *and* B *approach one another and meet, I say, the angle* BAD, *contained between the chord and the tangent, will be diminished* in infinitum, *and ultimately will vanish.*

For if that angle does not vanish, the arc ACB will contain with the tangent AD an angle equal to a rectilinear angle; and therefore the curvature at the point A will not be continued, which is against the supposition.

LEMMA VII.

The same things being supposed, I say that the ultimate ratio of the arc, chord, and tangent, any one to any other, is the ratio of equality, (Fig. 6.)

For while the point B approaches towards the point A, consider always AB and AD as produced to the remote points b and d, and parallel to the secant BD draw bd: and let the arc Acb be always similar to the arc ACB. Then, supposing the points A and B to coincide, the angle dAb will vanish, by the preceding Lemma; and therefore the right lines Ab, Ad, (which are always finite,) and the intermediate arc Acb, will coincide, and become equal among themselves. Wherefore the right lines AB, AD, and the intermediate arc ACB (which are always proportional to the former) will vanish, and ultimately acquire the ratio of equality. Q. E. D.

Cor. 1. Whence if through B (Fig. 7.) we draw BF parallel to the tangent, always cutting any right line AF passing through A in F, this line BF will be ultimately in the ratio of equality with the evanescent arc ACB; because, completing the parallelogram AFBD, it is always in a ratio of equality with AD.

Cor. 2. And if through B and A more right lines be drawn, as BE, BD, AF, AG, cutting the tangent AD and its parallel BF; the ultimate ratio of all the abscissas AD, AE, BF, BG, and of the chord and arc AB, any one to any other, will be the ratio of equality.

Cor. 3. And therefore in all our reasoning about ultimate ratios, we may freely use any one of those lines for any other.

LEMMA VIII.

If the right lines AR, BR, (Fig. 5.) *with the arc* ACB, *the chord* AB, *and the tangent* AD, *constitute three triangles* RAB, RACB, RAD, *and the points* A *and* B *approach and meet: I say, that the ultimate form of these evanescent triangles is that of similitude, and their ultimate ratio that of equality.*

For while the point B approaches towards the point A, consider always AB, AD, AR, as produced to the remote points b, d, and r, and rbd as drawn parallel to RD, and let the arc Acb be always similar to the arc ACB. Then supposing the points A and B to coincide, the angle bAd will vanish; and therefore the three triangles rAb, rAcb, rAd (which are always finite) will coincide, and on that account become both similar and equal. And therefore the triangles RAB, RACB, RAD, which are always similar and proportional to these, will ultimately become both similar and equal among themselves. Q. E. D.

COR. And hence in all our reasonings about ultimate ratios, we may indifferently use any one of those triangles for any other.

LEMMA IX.

If a right line AE, (Fig. 7.) *and a curve line* ABC, *both given by position, cut each other in a given angle* A; *and to that right line, in another given angle,* BD, CE *are ordinately applied, meeting the curve in* B, C; *and the points* B *and* C *together approach towards and meet in the point* A: *I say, that the areas of the triangles* ABD, ACE, *will ultimately be one to the other in the duplicate ratio of the sides.*

For while the points B, C approach towards the point A, suppose always AD to be produced to the remote points d and e, so as Ad, Ae may be proportional to AD, AE; and the ordinates db, ec, to be drawn parallel to the ordinates DB and EC, and meeting AB and AC produced in b and c. Let the curve Abc be similar to the curve ABC, and draw the right line Ag so as to touch both curves in A, and cut the ordinates DB, EC, db, ec, in F, G, f, g. Then, supposing the length Ae to remain the same, let the points B and C meet in the point A; and the angle cAg vanishing, the curvilinear areas Abd, Ace will coincide with the rectilinear areas Afd, Age; and therefore (by Lem. 5.) will be one to the other in the duplicate ratio of the sides Ad, Ae. But the areas ABD, ACE are always proportional to these areas; and so the sides AD, AE are to these sides. And therefore the areas ABD, ACE are ultimately one to the other in the duplicate ratio of the sides AD, AE. Q. E. D.

LEMMA X.

The spaces which a body describes by any finite force urging it, whether that force be determined and immutable, or be continually augmented or continually diminished, are in the very beginning of the motion one to the other in the duplicate ratio of the times.

Let the times be represented by the lines AD, AE, and the velocities generated in those times by the ordinates DB, EC. The spaces described with these velocities will be as the areas ABD, ACE, described by those ordinates, that is, at the very beginning of the motion (by Lem. 9.), in the duplicate ratio of the times AD, AE. Q. E. D.

Cor. 1. And hence one may easily infer, that the errors of bodies describing similar parts of similar figures in proportional times, are nearly in the duplicate ratio of the times in which they are generated; if these errors be generated by any equal forces similarly applied to the bodies, and measured by the distances of the bodies from those places of the similar figures, at which, without the action of those forces, the bodies would have arrived in those proportional times.

Cor. 2. But the errors that are generated by proportional forces, similarly applied to the bodies at similar parts of the similar figures, are as the forces and the squares of the times conjointly.

Cor. 3. The same thing is to be understood of any spaces whatsoever described by bodies urged with different forces; all which, in the very beginning of the motion, are as the forces and the squares of the times conjointly.

COR. 4. And therefore the forces are as the spaces described in the very beginning of the motion directly, and the squares of the times inversely.

COR. 5. And the squares of the times are as the spaces described directly, and the forces inversely.

SCHOLIUM.

If in comparing indetermined quantities of different sorts one with another, any one is said to be as any other directly or inversely, the meaning is, that the former is augmented or diminished in the same ratio with the latter, or with its reciprocal. And if any one is said to be as any other two or more directly or inversely, the meaning is, that the first is augmented or diminished in the ratio compounded of the ratios in which the others, or the reciprocals of the others, are augmented or diminished. As if A is said to be as B directly, and C directly, and D inversely, the meaning is, that A is augmented or diminished in the same ratio with $B \times C \times \frac{1}{D}$, that is to say, that A and $\frac{BC}{D}$ are one to the other in a given ratio.

LEMMA XI.

The evanescent subtense of the angle of contact, in all curves which at the point of contact have a finite curvature, is ultimately in the duplicate ratio of the subtense of the conterminous arc, (Fig. 8.)

CASE 1. Let AB be that arc, AD its tangent, BD the subtense of the angle of contact perpendicular on the tangent, AB the subtense of the arc. Draw BG perpendicular to the subtense AB, and AG to the tangent AD, meeting in G; then let the points D, B, and G approach to the points d, b, and g, and suppose J to be the ultimate intersection of the lines BG, AG, when the points D, B have come to A. It is evident that the distance GJ may be less than any assignable. But (from the nature of the circles passing through the points A, B, G, A, b, g) $AB^2 = AG \times BD$, and $Ab^2 = Ag \times bd$; and therefore the ratio of AB^2 to Ab^2 is compounded of the ratios of AG to Ag, and of BD to bd. But because GJ may be assumed of less length than any assignable, the ratio of AG to Ag may be such as to differ from the ratio of equality by less than any assignable difference; and therefore the ratio of AB^2 to Ab^2 may be such as to differ from the ratio of BD to bd by less than any assignable difference. Therefore, by Lem. 1. the ultimate ratio of AB^2 to Ab^2 is the same with the ultimate ratio of BD to bd. Q. E. D.

CASE 2. Now let BD be inclined to AD in any given angle, and the ultimate ratio of BD to bd will always be the same as before, and therefore the same with the ratio of AB^2 to Ab^2. Q. E. D.

CASE 3. And if we suppose the angle D not to be given, but that the right line BD converges to a given point, or is determined by any other condition whatever; nevertheless, the angles D, d, being determined by the same law, will always draw nearer to equality, and approach nearer to each other than by any assigned difference, and therefore, by Lem. 1. will at last be equal; and therefore the lines BD, bd are in the same ratio to each other as before. Q. E. D.

COR. 1. Therefore, since the tangents AD, Ad, the arcs AB, Ab, and their sines BC, bc, become ultimately equal to the chords AB, Ab, their squares will ultimately become as the subtenses BD, bd.

COR. 2. Their squares are also ultimately as the versed sines of the arcs, bisecting the chords, and converging to a given point. For those versed sines are as the subtenses BD, bd.

COR. 3. And therefore the versed sine is in the duplicate ratio of the time in which a body will describe the arc with a given velocity.

COR. 4. The rectilinear triangles ADB, Adb are ultimately in the triplicate ratio of the sides AD, Ad, and in a sesquiplicate ratio of the sides DB, db; as being in the ratio compounded of the sides AD to DB, and of Ad to db. So also the triangles ABC, Abc are ultimately in the triplicate ratio of the sides BC, bc. What I call the sesquiplicate ratio is the subduplicate of the triplicate, as being compounded of the simple and subduplicate ratio.

COR. 5. And because DB, db are ultimately parallel and in the duplicate ratio of the lines AD, Ad, the ultimate curvilinear areas ADB, Adb will be (by the nature of the parabola) two-thirds of the rectilinear triangles ADB, Adb; and the segments AB, Ab will be one-third of the same triangles. And thence those areas and those segments will be in the triplicate ratio as well of the tangents AD, Ad, as of the chords and arcs AB, Ab.

SCHOLIUM.

But we have all along supposed the angle of contact to be neither infinitely greater nor infinitely less than the angles of contact made by circles and their tangents; that is, that the curvature at the point A is neither infinitely small nor infinitely great, or that the interval AJ is of a finite magnitude. For DB may be taken as AD^3: in which case no circle can be drawn through the point A, between the tangent AD and the curve AB, and therefore the angle of contact will be infinitely less than those of circles. And by a like reasoning, if DB be made successively as AD^4, AD^5, AD^6, AD^7, &c. we shall have a series of angles of contact, proceeding *in infinitum*, wherein every succeeding term is infinitely less than the preceding. And if DB be made successively as AD^2, $AD^{\frac{3}{2}}$, $AD^{\frac{4}{3}}$,

$AD^{\frac{5}{4}}$, $AD^{\frac{6}{5}}$, $AD^{\frac{7}{6}}$, &c. we shall have another infinite series of angles of contact, the first of which is of the same sort with those of circles, the second infinitely greater, and every succeeding one infinitely greater than the preceding. But between any two of these angles another series of intermediate angles of contact may be interposed, proceeding both ways *in infinitum*, wherein every succeeding angle shall be infinitely greater or infinitely less than the preceding. As if between the terms AD^2 and AD^3 there were interposed the series $AD^{\frac{13}{6}}$, $AD^{\frac{11}{5}}$, $AD^{\frac{9}{4}}$, $AD^{\frac{7}{3}}$, $AD^{\frac{5}{2}}$, $AD^{\frac{8}{3}}$, $AD^{\frac{11}{4}}$, $AD^{\frac{14}{5}}$, $AD^{\frac{17}{6}}$, &c. And again, between any two angles of this series, a new series of intermediate angles may be interposed, differing from one another by infinite intervals. Nor is nature confined to any bounds.

Those things which have been demonstrated of curve lines, and the superficies which they comprehend, may be easily applied to the curve superficies and contents of solids. These Lemmas are premised to avoid the tediousness of deducing perplexed demonstrations *ad absurdum*, according to the method of the ancient geometers. For demonstrations are more contracted by the method of indivisibles: but because the hypothesis of indivisibles seems somewhat harsh, and therefore that method is reckoned less geometrical, I chose rather to reduce the demonstrastrations of the following propositions to the first and last sums and ratios of nascent and evanescent quantities, that is, to the limits of those sums and ratios; and so to premise, as short as I could, the demonstrations of those limits. For hereby the same thing is performed as by the method of indivisibles; and now those principles being demonstrated, we may use them with more safety. Therefore, if hereafter I should happen to consider quantities as made up of particles, or should use little curve lines for right ones, I would not be understood to mean indivisibles, but evanescent divisible quantities; not the sums and ratios of determinate parts, but always the limits of sums and ratios: and that the force of such demonstrations always depends on the method laid down in the foregoing Lemmas.

Perhaps it may be objected, that there is no ultimate proportion of evanescent quantities; because the proportion, before the quantities have vanished, is not the ultimate, and when they are vanished, is none. But by the same argument it may be alledged, that a body arriving at a certain place, and there stopping, has no ultimate velocity: because the velocity, before the body comes to the place, is not its ultimate velocity; when it has arrived, is none. But the answer is easy; for by the ultimate velocity is meant that with which the body is moved, neither before it arrives at its last place and the motion ceases, nor after, but at the very instant it arrives; that is, that velocity with which the body arrives at its last place, and with which the motion ceases. And in like manner, by the ultimate ratio of evanescent quantities is to be understood the ratio of the quantities not before they vanish, nor afterwards, but with which they vanish. In like manner the first ratio of nascent quantities is that with which they begin to be. And the first or last sum is that with which they begin and cease to be, (or to be augmented or diminished). There is a limit which the velocity at the end of the motion may attain, but not exceed. This is the ultimate velocity. And there is the like limit in all quantities and proportions that begin and cease to be. And since such limits are certain and definite, to determine the same is a problem strictly geometrical. But whatever is geometrical we may be allowed to use in determining and demonstrating any other thing that is likewise geometrical.

It may also be objected, that if the ultimate ratios of evanescent quantities are given, their ultimate magnitudes will be also given: and so all quantities will consist of indivisibles, which is contrary to what *Euclid* has demonstrated concerning incommensurables, in the 10th book of his Elements. But this objection is founded on a false supposition. For those ultimate ratios with which quantities vanish are not truly the ratios of ultimate quantities, but limits towards which the ratios of quantities decreasing without limit do always converge; and to which they approach nearer than by any given difference, but never go beyond, nor in effect attain to, till the quantities are diminished *in infinitum*. This thing will appear more evident in quantities infinitely great. If two quantities, whose difference is given, be augmented *in infinitum*, the ultimate-ratio of these quantities will be given, to wit, the ratio of equality; but it does not from thence follow, that the ultimate or greatest quantities themselves, whose ratio that is, will be given. Therefore if in what follows, for the sake of being more easily understood, I should happen to mention quantities as *least*, or *evanescent*, or *ultimate*, you are not to suppose that quantities of any determinate magnitude are meant, but such as are conceived to be always diminished without end.

SECTION II.

OF THE INVENTION OF CENTRIPETAL FORCES.

PROP. I.

The areas, which revolving bodies describe by radii drawn to an immoveable centre of force lie in the same immoveable planes, and are proportional to the times in which they are described, (Fig. 9.)

For suppose the time to be divided into equal parts, and in the first part of that time let the body by its innate force describe the right line AB. In the second part of that time, the same would, (by Law 1.), if not hindered, proceed directly to c, along the line Bc equal to AB; so that by the radii AS, BS, cS, drawn to the centre, the equal areas ABS, BSc, would be described. But when the body is arrived at B, suppose that a centripetal force acts at once with a great impulse, and, turning aside the body from the right line Bc, compels it afterwards to continue its motion along the right line BC. Draw cC parallel to BS meeting BC in C; and at the end of the second part of the time, the body (by Cor. 1. of the laws) will be found in C, in the same plane with the triangle ASB. Join SC, and, because SB and Cc are parallel, the triangle SBC will be equal to the triangle SBc, and therefore also to the triangle SAB. By the like argument, if the centripetal force act successively in C, D, E, &c. and make the body, in each single particle of time, to describe the right lines CD, DE, EF, &c. they will all lie in the same plane; and the triangle SCD will be equal to the triangle SBC, and SDE to SCD, and SEF to SDE. And therefore, in equal times, equal areas are described in one immoveable plane: and, by composition, any sums SADS, SAFS, of those areas, are one to the other as the times in which they are described. Now let the number of those triangles be augmented, and their breadth diminished *in infinitum*; and (by Cor. 4. Lem. 3.) their ultimate perimeter ADF will be a curve line: and therefore the centripetal force, by which the body is perpetually drawn back from the tangent of this curve, will act continually; and any described areas SADS, SAFS, which are always proportional to the times of description will, in this case also, be proportional to those times. Q. E. D.

COR. 1. The velocity of a body attracted towards an immoveable centre, in spaces void of resistance, is reciprocally as the perpendicular let fall from that centre on the right line that touches the orbit. For the velocities in those places A, B, C, D, E, are as the bases AB, BC, CD, DE, EF, of equal triangles; and these bases are reciprocally as the perpendiculars let fall upon them.

COR. 2. If the chords AB, BC of two arcs, successively described in equal times by the same body, in spaces void of resistance, be completed into a parallelogram ABCV, and the diagonal BV of this parallelogram, in the position which it ultimately acquires when those arcs are diminished *in infinitum*, be produced both ways, it will pass through the centre of force.

COR. 3. If the chords AB, BC, and DE, EF, of arcs described in equal times, in spaces void of resistance, be completed into the parallelograms ABCV, DEFZ; the forces in B and E are one to the other in the ultimate ratio of the diagonals BV, EZ, when those arcs are diminished *in infinitum*. For the motions BC and EF of the body (by Cor. 1. of the laws) are compounded of the motions Bc, BV, and Ff, EZ: but BV and EZ, which are equal to Cc and Ff, in the demonstration of this proposition, were generated by the impulses of the centripetal force in B and E, and are therefore proportional to those impulses.

COR. 4. The forces by which bodies, in spaces void of resistance, are drawn back from rectilinear motions, and turned into curvilinear orbits, are one to another as the *sagittæ* of arcs described in equal times; which *sagittæ* tend to the centre of force, and bisect the chords when those arcs are diminished to infinity. For such *sagittæ* are the halves of the diagonals mentioned in Cor. 3.

COR. 5. And therefore those forces are to the force of gravity as the said *sagittæ* to the *sagittæ* perpendicular to the horizon of those parabolic arcs which projectiles describe in the same time.

COR. 6. And the same things do all hold good (by Cor. 5. of the laws), when the planes in which the bodies are moved, together with the centres of force which are placed in those planes, are not at rest, but move uniformly forward in right lines.

PROP. II.

Every body that moves in any curve line described in a plane, and by a radius, drawn to a point either immoveable, or moving forward with an uniform rectilinear motion, describes about that point areas proportional to the times, is urged by a centripetal force directed to that point.

CASE 1. For every body that moves in a curve line, is (by Law 1.) turned aside from its rectilinear course by the action of some force that impels it. And that force by which the body is turned off from its rectilinear course, and is made to describe, in equal times, the equal least triangles SAB, SBC, SCD, &c. about the immoveable point S (by Prop. 40. Book I. Elem. and Law 2.) acts in the place B, according to the direction of a line parallel to cC, that is, in the direction of the line BS; and in the place C, according to the direction of a line parallel to dD, that is, in the direction of the line CS, &c.; and therefore acts always in the direction of lines tending to the immoveable point S. Q. E. D.

CASE 2. And (by Cor. 5 of the Laws) it is indifferent whether the superficies in which a body describes a curvilinear figure be quiescent, or moves together with the body, the figure described, and its point S, uniformly forwards in right lines.

COR. 1. In non-resisting spaces or mediums, if the areas be not proportional to the times, the forces are not directed to the point in which the radii meet; but deviate therefrom *in consequentia*, or towards the parts to which the motion is directed, if the description of the areas be accelerated; but *in antecedentia*, if retarded.

COR. 2. And even in resisting mediums, if the description of the areas be accelerated, the directions of the forces deviate from the point in which the radii meet, towards the parts to which the motion tends.

SCHOLIUM.

A body may be urged by a centripetal force compounded of several forces; in which case the meaning of the proposition is, that the force which results out of all tends to the point S. But if any force act perpetually in the direction of lines perpendicular to the described surface, this force will make the body to deviate from the plane of its motion; but will neither augment nor diminish the quantity of the described surface, and is therefore to be neglected in the composition of forces.

PROP. III.

Every body that, by a radius drawn to the centre of another body, howsoever moved, describes areas about that centre proportional to the times, is urged by a force compounded of a centripetal force tending to that other body, and of all the accelerative force by which that other body is impelled.

Let L represent the one, and T the other body; and (by Cor. 6. of the Laws) if both bodies be urged in the direction of parallel lines, by a new force equal and contrary to that by which the second body T is urged, the first body L will go on to describe about the other body T the same areas as before: but the force by which that other body T was urged will be now destroyed by an equal and contrary force; and therefore (by Law 1.) that other body T, now left to itself, will either rest, or move uniformly forward in a right line: and the first body L impelled by the difference of the forces, that is, by the force remaining, will go on to describe about the other body T areas proportional to the times. And therefore (by Theor. 2.) the difference of the forces is directed to the other body T as its centre. Q. E. D.

COR. 1. Hence if the one body L, by a radius drawn to the other body T, describe areas proportional to the times; and from the whole force, by which the first body L is urged (whether that force is simple, or, according to Cor. 2. of the Laws, compounded out of several forces), we subduct (by the same Cor.) that whole accelerative force by which the other body is urged; the whole remaining force by which the first body is urged will tend to the other body T, as its centre.

COR. 2. And, if these areas be proportional to the times nearly, the remaining force will tend to the other body T nearly.

COR. 3. And *vice versâ*, if the remaining force tend nearly to the other body T, those areas will be nearly proportional to the times.

COR. 4. If the body L, by a radius drawn to the other body T, describe areas which, compared with the times, are very unequal; and that other body T be either at rest, or move uniformly forward in a right line: the action of the centripetal force tending to that other body T is either none at all, or it is mixed and compounded with very powerful actions of other forces: and the whole force compounded of them all, if they be many, is directed to another (immoveable or moveable) centre. The same thing obtains, when the other body is moved by any motion whatsoever; provided that centripetal force is taken, which remains after subducting that whole force acting upon that other body T.

SCHOLIUM.

Because the equable description of areas indicates a tendency to a centre by that force with which the body is most affected, and by which it is drawn back from its rectilinear motion, and retained in its orbit; why may we not be allowed, in the following discourse, to use the equable description of areas as an index of a centre, about which all circular motion is performed in free spaces?

PROP. IV.

The centripetal forces of bodies, which by equable motions describe different circles, tend to the centres of the same circles; and are one to the other as the squares of the arcs described in equal times applied to the radii of the circles.

These forces tend to the centres of the circles, (by Prop. 2. and Cor. 2. Prop. 1.) and are one to another as the versed sines of the least arcs described in equal times, (by Cor. 4. Prop. 1.); that is, as the squares of the same arcs applied to the diameters of the circles, (by Lem. 7.): and therefore since those arcs are as arcs described in any equal times, and the diameters are as the radii, the forces will be as the squares of any arcs described in the same time applied to the radii of the circles. Q. E. D.

COR. 1. Therefore, since those arcs are as the velocities of the bodies, the centripetal forces are in a ratio compounded of the duplicate ratio of the velocities directly, and of the simple ratio of the radii inversely.

COR. 2. And since the periodic times are in a ratio compounded of the ratio of the radii directly, and the ratio of the velocities inversely, the centripetal forces are in a ratio compounded of the ratio of the radii directly, and the duplicate ratio of the periodic times inversely.

COR. 3. Whence if the periodic times be equal, and the velocities therefore as the radii, the centripetal forces will be also as the radii; and the contrary.

COR. 4. If the periodic times and the velocities be both in the subduplicate ratio of the radii, the centripetal forces will be equal among themselves; and the contrary.

Cor. 5. If the periodic times be as the radii, and therefore the velocities equal, the centripetal forces will be reciprocally as the radii; and the contrary.

Cor. 6. If the periodic times be in the sesquiplicate ratio of the radii, and therefore the velocities reciprocally in the subduplicate ratio of the radii, the centripetal forces will be in the duplicate ratio of the radii inversely; and the contrary.

Cor. 7. And universally, if the periodic time be as any power R^n of the radius R, and therefore the velocity reciprocally as the power R^{n-1} of the radius, the centripetal force will be reciprocally as the power R^{2n-1} of the radius; and the contrary.

Cor. 8. The same things all hold concerning the *times*, the *velocities*, and *forces* by which bodies describe the *similar* parts of any *similar* figures that have their centres in a similar position in those figures; as appears by applying the demonstration of the preceding cases to those. And the application is easy, by only substituting the equable description of areas in the place of equable motion, and using the distances of the bodies from the centres instead of the radii.

Cor. 9. From the same demonstration it likewise follows, that the arc which a body, uniformly revolving in a circle by means of a given centripetal force, describes in any time, is a mean proportional between the diameter of the circle, and the space which the same body falling by the same given force would descend through in the same given time.

SCHOLIUM.

The case of the 6th Corollary obtains in the celestial bodies, (as Sir *Christopher Wren*, Dr. *Hooke*, and Dr. *Halley* have severally observed); and therefore in what follows, I intend to treat more at large of those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centres.

Moreover, by means of the preceding proposition and its corollaries, we may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body by means of its gravity revolve in a circle concentric to the earth, this gravity is the centripetal force of that body. But, from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. 9. of this Prop.) And by such propositions, Mr. *Huygens*, in his excellent book *De Horologio Oscillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, be reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as that velocity and the number of reflections conjointly; that is, (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore, (if the polygon, by having its sides diminished *in infinitum*, coincide with the circle), as the square of the arc described in a given time applied to the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROP. V.

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre, (Fig. 10.)

Let the three right lines PT, TQV, VR touch the figure described in as many points P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, reciprocally proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C, of the perpendiculars draw AD, DBE, EC, at right angles, meeting in D and E; and the right lines TD, VE produced, will meet in S, the centre required.

For the perpendiculars let fall from the centre S on the tangents PT, QT, are reciprocally as the velocities of the bodies in the points P and Q, (by Cor. 1. Prop. 1.), and therefore, by construction, as the perpendiculars AP, BQ directly; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T, are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q. E. D.

PROP. VI.

In a space void of resistance, if a body revolve in any orbit about an immoveable centre, and in the least time describe any arc just then nascent; and the sagitta of that arc be supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force *in the middle of the arc will be as the sagitta directly and the square of the time inversely.*

For the *sagitta* in a given time is as the force (by Cor. 4. Prop. 1.); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the *sagitta* will be augmented in the duplicate of that ratio, (by Cor. 2 and 3. Lem. 11.), and therefore is as the force and the square of the time. Subduct on both sides the duplicate ratio of the time, and the force will be as the *sagitta* directly, and the square of the time inversely. Q. E. D.

And the same thing may also be easily demonstrated by Cor. 4. Lem. 10.

Cor. 1. If a body P revolving about the centre S, (Fig. 11.) describe a curve line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve, QR be drawn parallel to the distance SP, meeting the tangent in R; and QT be drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the solid $\frac{SP^2 \times QT^2}{QR}$, if the solid be taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the *sagitta* of double the arc QP, whose middle is P: and double the triangle SQP, or SP × QT is proportional to the time in which that double arc is described; and therefore may be used for the exponent of the time.

Cor. 2. By a like reasoning, the centripetal force is reciprocally as the solid $\frac{SY^2 \times QP^2}{QR}$; if SY be a perpendicular from the centre of force on PR the tangent of the orbit. For the rectangles SY × QP, and SP × QT are equal.

Cor. 3. If the orbit be either a circle, or if it touch or cut a circle concentrically, that is, contain with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be reciprocally as the solid $SY^2 \times PV$. For PV is $\frac{QP^2}{QR}$.

COR. 4. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular SY, by Cor. 1. Prop. 1.

COR. 5. Hence if any curvilinear figure APQ be given, and therein a point S be also given, to which a centripetal force is perpetually directed, that law of centripetal force may be found, by which the body P shall be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, shall describe the same by a perpetual revolution. That is, we are to find, by computation, either the solid $\frac{SP^2 \times QT^2}{QR}$, or the solid $SY^2 \times PV$, reciprocally proportional to this force. Examples of this we shall give in the following problems.

PROP. VII.

If a body revolve in the circumference of a circle, it is proposed to find the law of centripetal force directed to any given point, (Fig. 12.)

Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle at the preceding place. Through the point S draw the chord PV, and the diameter VA of the circle; join AP; and draw QT perpendicular to SP, which produced, may meet the tangent PR in Z; and lastly, through the point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, because of the similar triangles ZQR, ZTP, VPA, we shall have RP^2, that is, QRL to QT^2 as AV^2 to PV^2. And therefore $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2. Multiply those equals by $\frac{SP^2}{QR}$, and the points P and Q coinciding, for RL write PV; then we shall have $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. And therefore (by Cor. 1 and 5, Prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is, (because AV^2 is given,) reciprocally as the square of the distance or altitude SP, and the cube of the chord PV conjointly. Q. E. I.

The same otherwise.

On the tangent PR produced let fall the perpendicular SY; and (because of the similar triangles SYP, VPA) we shall have AV to PV as SP to SY, and therefore $\frac{SP \times PV}{AV} = SY$, and $\frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV$. And therefore (by Cor. 3 and 5, Prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is, (because AV is given,) reciprocally as $SP^2 \times PV^3$. Q. E. I.

Cor. 1. Hence if the given point S, to which the centripetal force always tends, be placed in the circumference of the circle, as at V, the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude SP.

Cor. 2. The force by which the body P in the circle APTV (Fig. 13.) revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \times SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction of this proposition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$: that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$; or (because of the similar triangles PSG, TPV) to SG^3.

COR. 3. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \times RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROP. VIII.

If a body move in the semi-circumference PQA; *it is proposed to find the law of the centripetal force tending to a point* S, *so remote, that all the lines* PS, RS *drawn thereto, may be taken for parallels,* (Fig. 14.)

From C, the centre of the semi-circle, let the semi-diameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM, PZT, and RZQ, we shall have CP^2 to PM^2 as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or, the points P, Q coinciding, to the rectangle $QR \times 2\,PM$. Therefore CP^2 is to PM^2 as $QR \times 2\,PM$ to QT^2; and

$\frac{QT^2}{QR} = \frac{2\,PM^3}{CP^2}$, and $\frac{QT^2 \times SP^2}{QR} = \frac{2\,PM^3 \times SP^2}{CP^2}$. And therefore (by Cor. 1 and 5, Prop. 6.) the centripetal force is reciprocally as $\frac{2\,PM^3 \times SP^2}{CP^2}$; that is, (neglecting the given ratio $\frac{2\,SP^2}{CP^2}$), reciprocally as PM^3. Q. E. I.

And the same thing is likewise easily inferred from the preceding proposition.

SCHOLIUM.

And by a like reasoning, a body shall be found to move in an ellipse, or even in an hyperbola, or parabola, by a centripetal force which is reciprocally as the cube of the ordinate directed to an infinitely remote centre of force.

PROP. IX.

If a body revolve in a spiral PQS, *cutting all the radii* SP, SQ, *&c. in a given angle; it is proposed to find the law of the centripetal force tending to the centre of that spiral,* (Fig. 15.)

Suppose the indefinitely small angle PSQ to be given; because, then, all the angles are given, the figure SPRQT will be given in species. Therefore the ratio $\frac{QT}{QR}$ is also given, and $\frac{QT^2}{QR}$ is as QT, that is, (because the figure is given in species), as SP. But if the angle PSQ is any way changed, the right line QR, subtending the angle of contact QPR (by Lem. 11.) will be changed in the duplicate ratio of PR or QT. Therefore the ratio $\frac{QT^2}{QR}$ remains the same as before, that is, as SP. And $\frac{QT^2 \times SP^2}{QR}$ is as SP^3, and therefore (by Cor. 1 and 5, Prop. 6.) the centripetal force is reciprocally as the cube of the distance SP. Q. E. I.

The same otherwise.

The perpendicular SY let fall upon the tangent, and the chord PV of the circle concentrically cutting the spiral, are in given ratios to the height SP; and therefore SP^3 is as $SY^2 \times PV$, that is (by Cor. 3 and 5, Prop. 6.), reciprocally as the centripetal force.

LEMMA XII.

All parallelograms circumscribed about any conjugate diameters of a given ellipse or hyperbola are equal among themselves.

This is demonstrated by the writers on conic sections.

PROP. X.

If a body revolve in an ellipse, it is proposed to find the law of the centripetal force tending to the centre of the ellipse, (Fig. 16.)

Suppose CA, CB to be semi-axes of the ellipse; GP, DK, conjugate diameters; PF, QT perpendiculars to those diameters; Qv an ordinate to the diameter GP; and if the parallelogram QvPR be completed, then (by the properties of the conic sections) the rectangle PvG will be to Qv^2 as PC^2 to CD^2; and (because of the similar triangles QvT, PCF) Qv^2 to QT^2 as PC^2 to PF^2; and, by composition, the ratio of PvG to QT^2 is compounded of the ratio of PC^2 to CD^2, and of the ratio of PC^2 to PF^2, that is, vG to $\frac{QT^2}{Pv}$ as PC^2 to $\frac{CD^2 \times PF^2}{PC^2}$. Put QR for Pv, and (by Lem. 12.) $BC \times CA$ for $CD \times PF$; also (the points P and Q coinciding,) 2 PC for vG; and multiplying, the extremes and means together, we shall have $\frac{QT^2 \times PC^2}{QR}$ equal to $\frac{2\,BC^2 \times CA^2}{PC}$. Therefore, (by Cor. 5. Prop. 6.) the centripetal force is reciprocally as $\frac{2\,BC^2 \times CA^2}{PC}$; that is, (because $2\,BC^2 \times CA^2$ is given), reciprocally as $\frac{1}{PC}$; that is, directly as the distance PC. Q. E. I.

The same otherwise.

In the right line PG on the other side of the point T, take the point u so that Tu may be equal to Tv; then take nV, such as shall be to vG as DC^2 to PC^2. And because Qv^2 is to PvG as DC^2 to PC^2, (by the conic sections,) we shall have $Qv^2 = Pv \times uV$. Add the rectangle uPv to both sides, and the square of the chord of the arc PQ will be equal to the rectangle VPv; and therefore a circle which touches the conic section in P, and passes through the point Q, will pass also through the point V. Now let the points P and Q meet, and the ratio of uV to vG, which is the same with the ratio of DC^2 to PC^2, will become the ratio of PV to PG, or PV to 2 PC; and therefore PV will be equal to $\frac{2\,DC^2}{PC}$. And therefore the force by which the body P revolves in the ellipse will be reciprocally as $\frac{2\,DC^2}{PC} \times PF^2$, (by Cor. 3. Prop. 6.); that is (because $2\,DC^2 \times PF^2$ is given), directly as PC. Q. E. I.

COR. 1. And therefore the force is as the distance of the body from the centre of the ellipse; and *vice versâ*, if the force is as the distance, the body will move in an ellipse whose centre coincides with the centre of force, or perhaps in a circle into which the ellipse may degenerate.

COR. 2. And the periodic times of the revolutions made in all ellipses whatsoever about the same centre will be equal. For those times in similar ellipses will be equal, (by Cor. 3 and 8, Prop. 4.); but in ellipses that have their greater axis common, they are one to another as the whole areas of the ellipses directly, and the parts of the areas described in the same time inversely; that is, as the lesser axes directly, and the velocities of the bodies in their principal vertices inversely; that is, as those lesser axes directly, and the ordinates to the same point of the common axis inversely; and therefore (because of the equality of the direct and inverse ratios) in the ratio of equality.

SCHOLIUM.

If the ellipse, by having its centre removed to an infinite distance, degenerate into a parabola, the body will move in this parabola; and the force, now tending to a centre infinitely remote, will become equable. Which is *Galileo's* theorem. And if the parabolic section of the cone (by changing the inclination of the cutting plane to the cone) degenerates into an hyperbola, the body will move in the perimeter of this hyperbola, having its centripetal force changed into a centrifugal force. And in like manner as in the circle, or in the ellipse, if the forces be directed to the centre of the figure placed in the abscissa, those forces, by increasing or diminishing the ordinates in any given ratio, or even by changing the angle of the inclination of the ordinates to the abscissa, are always augmented or diminished in the ratio of the distances from the centre; provided the periodic times remain equal: so also in all figures whatsoever, if the ordinates be augmented or diminished in any given ratio, or their inclination be any way changed, the periodic time remaining the same, the forces directed to any centre placed in the abscissa are in the several ordinates augmented or diminished in the ratio of the distances from the centre.

follows, I intend to treat more at large of those things which relate to centripetal force decreasing in a duplicate ratio of the distances from the centres.

Moreover, by means of the preceding proposition and its corollaries, we may discover the proportion of a centripetal force to any other known force, such as that of gravity. For if a body by means of its gravity revolve in a circle concentric to the earth, this gravity is the centripetal force of that body. But, from the descent of heavy bodies, the time of one entire revolution, as well as the arc described in any given time, is given (by Cor. 9. of this Prop.) And by such propositions, Mr. *Huygens*, in his excellent book *De Horologio Oscillatorio*, has compared the force of gravity with the centrifugal forces of revolving bodies.

The preceding proposition may be likewise demonstrated after this manner. In any circle suppose a polygon to be inscribed of any number of sides. And if a body, moved with a given velocity along the sides of the polygon, be reflected from the circle at the several angular points, the force, with which at every reflection it strikes the circle, will be as its velocity: and therefore the sum of the forces, in a given time, will be as that velocity and the number of reflections conjointly; that is, (if the species of the polygon be given), as the length described in that given time, and increased or diminished in the ratio of the same length to the radius of the circle; that is, as the square of that length applied to the radius; and therefore, (if the polygon, by having its sides diminished *in infinitum*, coincide with the circle), as the square of the arc described in a given time applied to the radius. This is the centrifugal force, with which the body impels the circle; and to which the contrary force, wherewith the circle continually repels the body towards the centre, is equal.

PROP. V.

There being given, in any places, the velocity with which a body describes a given figure, by means of forces directed to some common centre: to find that centre, (Fig. 10.)

Let the three right lines PT, TQV, VR touch the figure described in as many points P, Q, R, and meet in T and V. On the tangents erect the perpendiculars PA, QB, RC, reciprocally proportional to the velocities of the body in the points P, Q, R, from which the perpendiculars were raised; that is, so that PA may be to QB as the velocity in Q to the velocity in P, and QB to RC as the velocity in R to the velocity in Q. Through the ends A, B, C, of the perpendiculars draw AD, DBE, EC, at right angles, meeting in D and E; and the right lines TD, VE produced, will meet in S, the centre required.

For the perpendiculars let fall from the centre S on the tangents PT, QT, are reciprocally as the velocities of the bodies in the points P and Q, (by Cor. 1. Prop. 1.), and therefore, by construction, as the perpendiculars AP, BQ directly; that is, as the perpendiculars let fall from the point D on the tangents. Whence it is easy to infer that the points S, D, T, are in one right line. And by the like argument the points S, E, V are also in one right line; and therefore the centre S is in the point where the right lines TD, VE meet. Q. E. D.

PROP. VI.

In a space void of resistance, if a body revolve in any orbit about an immoveable centre, and in the least time describe any arc just then nascent; and the sagitta of that arc be supposed to be drawn bisecting the chord, and produced passing through the centre of force: the centripetal force *in the middle of the arc will be as the sagitta directly and the square of the time inversely.*

For the *sagitta* in a given time is as the force (by Cor. 4. Prop. 1.); and augmenting the time in any ratio, because the arc will be augmented in the same ratio, the *sagitta* will be augmented in the duplicate of that ratio, (by Cor. 2 and 3. Lem. 11.), and therefore is as the force and the square of the time. Subduct on both sides the duplicate ratio of the time, and the force will be as the *sagitta* directly, and the square of the time inversely. Q. E. D.

And the same thing may also be easily demonstrated by Cor. 4. Lem. 10.

COR. 1. If a body P revolving about the centre S, (Fig. 11.) describe a curve line APQ, which a right line ZPR touches in any point P; and from any other point Q of the curve, QR be drawn parallel to the distance SP, meeting the tangent in R; and QT be drawn perpendicular to the distance SP; the centripetal force will be reciprocally as the solid $\frac{SP^2 \times QT^2}{QR}$, if the solid be taken of that magnitude which it ultimately acquires when the points P and Q coincide. For QR is equal to the *sagitta* of double the arc QP, whose middle is P: and double the triangle SQP, or SP × QT is proportional to the time in which that double arc is described; and therefore may be used for the exponent of the time.

COR. 2. By a like reasoning, the centripetal force is reciprocally as the solid $\frac{SY^2 \times QP^2}{QR}$; if SY be a perpendicular from the centre of force on PR the tangent of the orbit. For the rectangles SY × QP, and SP × QT are equal.

COR. 3. If the orbit be either a circle, or if it touch or cut a circle concentrically, that is, contain with a circle the least angle of contact or section, having the same curvature and the same radius of curvature at the point P; and if PV be a chord of this circle, drawn from the body through the centre of force; the centripetal force will be reciprocally as the solid $SY^2 \times PV$. For PV is $\frac{QP^2}{QR}$.

COR. 4. The same things being supposed, the centripetal force is as the square of the velocity directly, and that chord inversely. For the velocity is reciprocally as the perpendicular SY, by Cor. 1. Prop. 1.

COR. 5. Hence if any curvilinear figure APQ be given, and therein a point S be also given, to which a centripetal force is perpetually directed, that law of centripetal force may be found, by which the body P shall be continually drawn back from a rectilinear course, and, being detained in the perimeter of that figure, shall describe the same by a perpetual revolution. That is, we are to find, by computation, either the solid $\frac{SP^2 \times QT^2}{QR}$, or the solid $SY^2 \times PV$, reciprocally proportional to this force. Examples of this we shall give in the following problems.

PROP. VII.

If a body revolve in the circumference of a circle, it is proposed to find the law of centripetal force directed to any given point, (Fig. 12.)

Let VQPA be the circumference of the circle; S the given point to which as to a centre the force tends; P the body moving in the circumference; Q the next place into which it is to move; and PRZ the tangent of the circle at the preceding place. Through the point S draw the chord PV, and the diameter VA of the circle; join AP; and draw QT perpendicular to SP, which produced, may meet the tangent PR in Z; and lastly, through the point Q, draw LR parallel to SP, meeting the circle in L, and the tangent PZ in R. And, because of the similar triangles ZQR, ZTP, VPA, we shall have RP^2, that is, QRL to QT^2 as AV^2 to PV^2. And therefore $\frac{QRL \times PV^2}{AV^2}$ is equal to QT^2. Multiply those equals by $\frac{SP^2}{QR}$, and the points P and Q coinciding, for RL write PV; then we shall have $\frac{SP^2 \times PV^3}{AV^2} = \frac{SP^2 \times QT^2}{QR}$. And therefore (by Cor. 1 and 5, Prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is, (because AV^2 is given,) reciprocally as the square of the distance or altitude SP, and the cube of the chord PV conjointly. Q. E. I.

The same otherwise.

On the tangent PR produced let fall the perpendicular SY; and (because of the similar triangles SYP, VPA) we shall have AV to PV as SP to SY, and therefore $\frac{SP \times PV}{AV} = SY$, and $\frac{SP^2 \times PV^3}{AV^2} = SY^2 \times PV$. And therefore (by Cor. 3 and 5, Prop. 6.) the centripetal force is reciprocally as $\frac{SP^2 \times PV^3}{AV^2}$; that is, (because AV is given,) reciprocally as $SP^2 \times PV^3$. Q. E. I.

Cor. 1. Hence if the given point S, to which the centripetal force always tends, be placed in the circumference of the circle, as at V, the centripetal force will be reciprocally as the quadrato-cube (or fifth power) of the altitude SP.

Cor. 2. The force by which the body P in the circle APTV (Fig. 13.) revolves about the centre of force S is to the force by which the same body P may revolve in the same circle, and in the same periodic time, about any other centre of force R, as $RP^2 \times SP$ to the cube of the right line SG, which from the first centre of force S is drawn parallel to the distance PR of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For by the construction of this proposition, the former force is to the latter as $RP^2 \times PT^3$ to $SP^2 \times PV^3$: that is, as $SP \times RP^2$ to $\frac{SP^3 \times PV^3}{PT^3}$; or (because of the similar triangles PSG, TPV) to SG^3.

COR. 3. The force by which the body P in any orbit revolves about the centre of force S, is to the force by which the same body may revolve in the same orbit, and the same periodic time, about any other centre of force R, as the solid $SP \times RP^2$, contained under the distance of the body from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of force S, and the square of its distance from the second centre of force R, to the cube of the right line SG, drawn from the first centre of force S, parallel to the distance RP of the body from the second centre of force R, meeting the tangent PG of the orbit in G. For the force in this orbit at any point P is the same as in a circle of the same curvature.

PROP. VIII.

If a body move in the semi-circumference PQA; *it is proposed to find the law of the centripetal force tending to a point* S, *so remote, that all the lines* PS, RS *drawn thereto, may be taken for parallels*, (Fig. 14.)

From C, the centre of the semi-circle, let the semi-diameter CA be drawn, cutting the parallels at right angles in M and N, and join CP. Because of the similar triangles CPM, PZT, and RZQ, we shall have CP^2 to PM^2 as PR^2 to QT^2; and, from the nature of the circle, PR^2 is equal to the rectangle $QR \times \overline{RN + QN}$, or, the points P, Q coinciding, to the rectangle $QR \times 2\,PM$. Therefore CP^2 is to PM^2 as $QR \times 2\,PM$ to QT^2; and

P and Q coincide. But the *lineola* QR in a given time is as the generating centripetal force; that is (by supposition), reciprocally as SP^2. And therefore $\frac{QT^2}{QR}$ is as $QT^2 \times SP^2$; that is, the latus rectum L is in the duplicate ratio of the area $QT \times SP$. Q. E. D.

Cor. Hence the whole area of the ellipse, and the rectangle under the axes, which is proportional to it, is in the ratio compounded of the subduplicate ratio of the latus rectum, and the ratio of the periodic time. For the whole area is as the area $QT \times SP$, described in a given time, multiplied by the periodic time.

PROP. XV.

The same things being supposed, I say, that the periodic times in ellipses are in the sesquiplicate ratio of their greater axes.

For the lesser axis is a mean proportional between the greater axis and the latus rectum; and, therefore, the rectangle under the axes is in the ratio compounded of the subduplicate ratio of the latus rectum and the sesquiplicate ratio of the greater axis. But this rectangle (by Cor. 3. Prop. 14.) is in a ratio compounded of the subduplicate ratio of the latus rectum, and the ratio of the periodic time. Subduct from both sides the subduplicate ratio of the latus rectum, and there will remain the sesquiplicate ratio of the greater axis, equal to the ratio of the periodic time. Q. E. D.

Cor. Therefore the periodic times in ellipses are the same as in circles whose diameters are equal to the greater axes of the ellipses.

PROP. XVI.

The same things being supposed, and right lines being drawn to the bodies that shall touch the orbits, and perpendiculars *being let fall on those tangents from the common focus; I say, that the velocities of the bodies are in a ratio compounded of the ratio of the perpendiculars inversely, and the subduplicate ratio of the principal latera recta directly,* (Fig. 22.)

From the focus S draw SY perpendicular to the tangent PR, and the velocity of the body P will be reciprocally in the subduplicate ratio of the quantity $\frac{SY^2}{L}$. For that velocity is as the infinitely small arc PQ described in a given moment of time, that is (by Lem. 7.) as the tangent PR; that is, (because of the proportionals PR to QT, and SP to SY), as $\frac{SP \times QT}{SY}$; or as SY reciprocally, and $SP \times QT$ directly; but $SP \times QT$ is as the area described in the given time, that is (by Prop. 14.) in the subduplicate ratio of the latus rectum. Q. E. D.

COR. 1. The principal latera recta are in a ratio compounded of the duplicate ratio of the perpendiculars and the duplicate ratio of the velocities.

COR. 2. The velocities of bodies, in their greatest and least distances from the common focus, are in the ratio compounded of the ratio of the distances inversely, and the subduplicate ratio of the principal latera recta directly. For those perpendiculars are now the distances.

COR. 3. And therefore the velocity in a conic section, at its greatest or least distance from the focus, is to the velocity in a circle, at the same distance from the centre, in the subduplicate ratio of the principal latus rectum to the double of that distance.

COR. 4. The velocities of the bodies revolving in ellipses, at their mean distances from the common focus, are the same as those of bodies revolving in circles, at the same distances; that is (by Cor. 6. Prop. 4.) reciprocally in the subduplicate ratio of the distances. For the perpendiculars are now the lesser semi-axes, and these are as mean proportionals between the distances and the latera recta. Let this ratio inversely be compounded with the subduplicate ratio of the latera recta directly, and we shall have the subduplicate ratio of the distance inversely.

COR. 5. In the same figure, or even in different figures, whose principal latera recta are equal, the velocity of a body is reciprocally as the perpendicular let fall from the focus on the tangent.

COR. 6. In a parabola, the velocity is reciprocally in the subduplicate ratio of the distance of the body from the focus of the figure; it is more variable in the ellipse, and less in the hyperbola, than according to this ratio. For (by Cor. 2. Lem. 14.) the perpendicular let fall from the focus on the tangent of a parabola is in the subduplicate ratio of the distance. In the hyperbola the perpendicular is less variable; in the ellipse more.

COR. 7. In a parabola, the velocity of a body at any distance from the focus is to the velocity of a body revolving in a circle, at the same distance from the centre, in the subduplicate ratio of the number 2 to 1; in the ellipse it is less, and in the hyperbola greater, than according to this ratio. For (by Cor. 2. of this Prop.) the velocity at the vertex of a parabola is in this ratio, and (by Cor. 6. of this Prop. and Prop. 4.) the same proportion holds in all distances. And hence, also, in a parabola, the velocity is every where equal to the velocity of a body revolving in a circle at half the distance; in the ellipse it is less, and in the hyperbola greater.

COR. 8. The velocity of a body revolving in any conic section is to the velocity of a body revolving in a circle, at the distance of half the principal latus rectum of the section, as that distance to the perpendicular let fall from the focus on the tangent of the section. This appears from Cor. 5.

COR. 9. Wherefore since (by Cor. 6. Prop. 4.) the velocity of a body revolving in this circle is to the velocity of another body revolving in any other circle reciprocally in the subduplicate ratio of the distances; therefore, *ex æquo*, the velocity of a body revolving in a conic section will be to the velocity of a body revolving in a circle at the same distance as a mean proportional between that common distance, and half the principal latus rectum of the section, to the perpendicular let fall from the common focus upon the tangent of the section.

PROP. XVII.

Supposing the centripetal force to be reciprocally proportional to the squares of the distances of places from the centre, and that the absolute quantity of that force is known; it is required to determine the line which a body will describe that is let go from a given place with a given velocity in the direction of a given right line.

Let the centripetal force tending to the point S (Fig. 23.) be such, as will make the body p revolve in any given orbit pq; and suppose the velocity of this body in the place p is known. Then from the place P suppose the body P to be let go with a given velocity in the direction of the line PR; but by virtue of a centripetal force to be immediately turned aside from that right line into the conic section PQ. This the right line PR will therefore touch in P. Suppose likewise that the right line pr touches the orbit pq in p; and if from S you suppose perpendiculars let fall on those tangents, the principal latus rectum of the conic section (by Cor. 1. Prop. 16.) will be to the principal latus rectum of that orbit in a ratio compounded of the duplicate ratio of the perpendiculars, and the duplicate ratio of the velocities; and is therefore given. Let this latus rectum be L: the focus S of the conic section is also given. Let the angle RPH be the complement of the angle RPS to two right; and the line PH, in which the other focus H is placed, is given by position. Let fall SK perpendicular on PH, and erect the conjugate semi-axis BC; this done, we shall have $SP^2 - 2KPH + PH^2 = SH^2 = 4CH^2 = 4BH^2 - 4BC^2 = \overline{SP + PH}^2 - L \times \overline{SP + PH} = SP^2 + 2SPH + PH^2 - L \times \overline{SP + PH}$. Add on both sides $2KPH - SP^2 - PH^2 + L \times \overline{SP + PH}$, and we shall have $L \times \overline{SP + PH} = 2SPH + 2KPH$, or $SP + PH$ to PH, as $2SP + 2KP$ to L. Whence PH is given both in length and position. That is, if the velocity of the body in P is such that the latus rectum L is less than $2SP + 2KP$, PH will lie on the same side of the tangent PR with the line SP; and therefore the figure will be an ellipse, which from the given foci S, H, and the principal axis $SP + PH$, is given also. But if the velocity of the body be so great, that the latus rectum L becomes equal to $2PS + 2KP$, the length PH will be infinite; and therefore, the figure will be a parabola, which has its axis SH parallel to the line PK, and is thence given. But if the body go from its place P with a yet greater velocity, the length PH is to be taken on the other side the tangent; and so the tangent passing between the foci, the figure will be an hyperbola having its principal axis equal to the difference of the lines SP and PH, and thence is given. For if the body, in these cases, revolve in a conic section so found, it is demonstrated in Prop. 11, 12, and 13, that the centripetal force will be reciprocally as the square of the distance of the body from the centre of force S; and therefore we have rightly determined the line PQ, which a body let go from a given place P with a given velocity, and in the direction of the right line PR given by position, would describe with such a force. Q. E. F.

COR. 1. Hence in every conic section, from the principal vertex D, the latus rectum L, and the focus S given, the other focus H is given, by taking DH to DS as the latus rectum to the difference between the latus rectum and 4 DS. For the proportion, SP+PH to PH as 2 PS+2 KP to L, becomes, in the case of this corollary, DS+DH to DH as 4 DS to L, and by division DS to DH as 4 DS — L to L.

Cor. 2. Whence if the velocity of a body in the principal vertex D be given, the orbit may be readily found; to wit, by taking its latus rectum to twice the distance DS, in the duplicate ratio of this given velocity to the velocity of a body revolving in a circle at the distance DS (by Cor. 3. Prop. 16.) and then taking DH to DS as the latus rectum to the difference between the latus rectum and 4 DS.

Cor. 3. Hence also if a body move in any conic section, and be forced out of its orbit by any impulse, you may discover the orbit in which it will afterwards pursue its course. For by compounding the proper motion of the body with that motion, which the impulse alone would generate, you will have the motion with which the body will go off from a given place of impulse in the direction of a right line given in position.

Cor. 4. And if that body be continually disturbed by the action of some foreign force, we may nearly know its course, by collecting the changes which that force introduces in some points, and estimating the continual changes it will undergo in the intermediate places, from the analogy that appears in the progress of the series.

SCHOLIUM.

If a body P, (Fig. 24.), by means of a centripetal force tending to any given point R, move in the perimeter of any given conic section whose centre is C; and the law of the centripetal force is required: draw CG parallel to the radius RP, and meeting the tangent PG of the orbit in

G; and the force required (by Cor. 1. and Schol. Prop. 10. and Cor. 3. Prop. 7.) will be as $\frac{CG^3}{RP^2}$.

And so far concerning the motion of bodies in curve lines. But it may also come to pass that a moving body shall ascend or descend in a right line; and I shall now go on to explain what belongs to such kind of motions.

SECTION VII.

CONCERNING THE RECTILINEAR ASCENT AND DESCENT OF BODIES.

PROP. XXXII.

Supposing that the centripetal force be reciprocally proportional to the square of the distance of the places from the centre; it is required to define the spaces which a body, falling directly, describes in given times.

CASE 1. If the body do not fall perpendicularly, it will (by Cor. 1. Prop. 13.) describe some conic section whose focus is placed in the centre of force. Suppose that conic section to be ARPB, (Fig. 25.), and its focus S. And, first, if the figure be an ellipse, upon the greater axis thereof AB describe the semi-circle ADB, and let the right line DPC pass through the falling body, making right angles with the axis; and drawing DS, PS, the area ASD will be proportional to the area ASP, and therefore also to the time. The axis AB still remaining the same, let the breadth of the ellipse be perpetually diminished, and the area ASD will always remain proportional to the time. Suppose that breadth to be diminished *in infinitum*;

and the orbit APB in that case coinciding with the axis AB, and the focus S with the extreme point of the axis B, the body will descend in the right line AC, and the area ABD will become proportional to the time. Wherefore the space AC will be given which the body describes in a given time by its perpendicular fall from the place A, if the area ABD be taken proportional to the time, and from the point D the right line DC be let fall perpendicular on the right line AB. Q. E. I.

Case 2. If the figure RPB be an hyperbola (Fig. 26.), on the same principal diameter AB describe the rectangular hyperbola BED; and because the areas CSP, CBfP, SPfB, are severally to the several areas CSD, CBED, SDEB, in the given ratio of the heights CP, CD, and the area SPfB is proportional to the time in which the body P will move through the arc PfB, the area SDEB will be also proportional to that time. Let the latus rectum of the hyperbola RPB be diminished *in infinitum*, the latus transversum remaining the same; and the arc PB will come to coincide with the right line CB, and the focus S with the vertex B, and the right line SD with the right line BD. And therefore the area BDEB will be proportional to the time in which the body C, by its perpendicular descent, describes the line CB. Q. E. I.

Case 3. And by the like argument, if the figure RPB be a parabola, (Fig. 27.), and to the same principal vertex B another parabola BED be described, that may always remain given while the former parabola in whose perimeter the body P moves, by having its latus rectum diminished and reduced to nothing, comes to coincide with the line CB, the parabolic segment BDEB will be proportional to the time in which that body P or C will descend to the centre S or B. Q. E. I.

PROP. XXXIII.

The things above found being supposed, I say, that the velocity of a falling body in any place C *is to the velocity of a body, describing a circle about the centre* B *at the distance* BC, *in the subduplicate ratio of* AC, *the distance of the body from the more remote vertex* A *of the circle or rectangular hyperbola, to* ½ AB, *the principal semi-diameter of the figure,* (Fig. 28.)

Let AB, the common diameter of both figures RPB, DEB, be bisected in O; and draw the right line PT that may touch the figure RPB in P, and likewise cut that common diameter AB (produced if necessary) in T; and let SY be perpendicular to this line, and BQ to this diameter, and suppose the latus rectum of the figure RPB to be L. From Cor. 9. Prop. 16. it is manifest that the velocity of a body, moving in the line RPB about the centre S, in any place P, is to the velocity of a body describing a circle about the same centre, at the distance SP, in the subduplicate ratio of the rectangle ½ L × SP to SY^2. For by the properties of the conic sections ACB is to CP^2 as 2AO to L, and therefore $\frac{2CP^2 \times AO}{ACB}$ is equal to L. Therefore those velocities are to each other in the subduplicate ratio of $\frac{CP^2 \times AO \times SP}{ACB}$ to SY^2. Moreover, by the properties of the conic sections, CO is to BO as BO to TO, and (by composition or division) as CB to BT. Whence (by division or composition) BO— or + CO will be to BO as CT to BT, that is, AC will be to AO as CP to BQ; and therefore $\frac{CP^2 \times AO \times SP}{ACB}$ is equal to $\frac{BQ^2 \times AC \times SP}{AO \times BC}$. Now suppose CP, the breadth of the figure RPB, to be diminished *in infinitum*, so that the point P may come to coincide with the point C, and the point S with the point B, and the line SP with the line BC, and the line SY with the line BQ; and the velocity of the body now descending perpendicularly in the line CB will be to the velocity of a body describing a circle about the centre B, at the distance BC, in the subduplicate ratio of $\frac{BQ^2 \times AC \times SP}{AO \times BC}$ to SY^2, that is (neglecting the ratios of equality of SP to BC, and BQ^2 to SY^2), in the subduplicate ratio of AC to AO, or $\frac{1}{2}$ AB. Q. E. D.

COR. 1. When the points B and S come to coincide, TC will become to TS as AC to AO.

COR. 2. A body revolving in any circle at a given distance from the centre, by its motion converted upwards, will ascend to double its distance from the centre.

PROP. XXXIV.

If the figure BED *be a parabola, I say, that the velocity of a falling body in any place* C *is equal to the velocity by which a body may uniformly describe a circle about the centre* B *at half the interval* BC, (Fig. 29.)

For (by Cor. 7. Prop. 16.) the velocity of a body describing a parabola RPB about the centre S, in any place P, is equal to the velocity of a body uniformly describing a circle about the same centre S at half the interval SP. Let the breadth CP of the parabola be diminished *in infinitum*, so that the parabolic arc PfB may come to coincide with the right line CB, the centre S with the vertex B, and the interval SP with the interval BC, and the proposition will be manifest. Q. E. D.

PROP. XXXV.

The same things supposed, I say, that the area of the figure DES, *described by the indefinite radius* SD, *is equal to* *the area which a body with a radius equal to half the latus rectum of the figure* DES, *by uniformly revolving about the centre* S, *may describe in the same time*, (Fig. 30.)

For suppose a body C in the smallest moment of time, to describe in falling the infinitely little line Cc, while another body K, uniformly revolving about the centre S in the circle OKk, describes the arc Kk. Erect the perpendiculars CD, cd, meeting the figure DES in D, d. Join SD, Sd, SK, Sk, and draw Dd meeting the axis AS in T, and thereon let fall the perpendicular SY.

Case 1. If the figure DES be a circle, or a rectangular hyperbola, bisect its tranverse diameter AS in O, and SO will be half the latus rectum. And because TC is to TD as Cc to Dd, and TD to TS as CD to SY; *ex æquo* TC will be to TS as CD × Cc to SY × Dd. But (by Cor. 1. Prop. 33.) TC is to TS as AC to AO; to wit, if in the coalescence of the points D, d, the ultimate ratios of the lines be taken. Wherefore AC is to AO or SK as CD × Cc to SY × Dd. Farther, the velocity of the descending body in C is to the velocity of a body describing a circle about the centre S, at the interval SC, in the subduplicate ratio of AC to AO or SK (by Prop. 33.); and this velocity is to the velocity of a body describing the circle OKk in the subduplicate ratio of SK to SC (by Cor. 6. Prop. 4.); and, *ex æquo*, the first velocity to the last, that is, the little line Cc to the arc Kk, in the subduplicate ratio of AC to SC, that is, in the ratio of AC to CD. Wherefore CD × Cc is equal to AC × Kk, and consequently AC to SK as AC × Kk to SY × Dd, and thence SK × Kk equal to SY × Dd, and $\frac{1}{2}$ SK × Kk equal to $\frac{1}{2}$ SY × Dd, that is, the area KSk equal to the area SDd. Therefore in every moment of time two equal particles, KSk and SDd, of areas are generated, which, if their magnitude be diminished, and their number increased *in infinitum*, obtain the ratio of equality, and consequently (by Cor. Lem. 4.) the whole areas together generated are always equal. Q. E. D.

CASE 2. But if the figure DES (Fig. 31.) be a parabola, we shall find, as above, CD × Cc to SY × Dd as TC to TS, that is, as 2 to 1; and that therefore ¼ CD × Cc is equal to ½ SY × Dd. But the velocity of the falling body in C is equal to the velocity with which a circle may be uniformly described at the interval ½ SC (by Prop. 34.). And this velocity to the velocity with which a circle may be described with the radius SK, that is, the little line Cc to the arc Kk, is (by Cor. 5. Prop. 4.) in the subduplicate ratio of SK to ½ SC; that is, in the ratio of SK to ½ CD. Wherefore ½ SK × Kk is equal to ¼ CD × Cc, and therefore equal to ½ SY × Dd; that is, the area KSk is equal to the area SDd, as above. Q. E. D.

PROP. XXXVI.

To determine the times of the descent of a body falling from a given place A, (Fig. 32.)

Upon the diameter AS, the distance of the body from the centre at the beginning, describe the semi-circle ADS, as likewise the semi-circle OKH equal thereto, about the centre S. From any place C of the body erect the ordinate CD. Join SD, and make the sector OSK equal to the area ASD. It is evident (by Prop. 35.) that the body in falling will describe the space AC in the same time in which another body, uniformly revolving about the centre S, may describe the arc OK. Q. E. F.

PROP. XXXVII.

To define the times of the ascent or descent of a body projected upwards or downwards from a given place, (Fig. 33.)

Suppose the body to go off from the given place G, in the direction of the line GS, with any velocity. In the duplicate ratio of this velocity to the uniform velocity in a circle, with which the body may revolve about the centre S at the given interval SG, take GA to $\frac{1}{2}$ AS. If that ratio be the same as of the number 2 to 1, the point A is infinitely remote; in which case a parabola is to be described with any latus rectum to the vertex S, and axis SG; as appears by Prop. 34. But if that ratio be less or greater than the ratio of 2 to 1, in the former case a circle, in the latter a rectangular hyperbola, is to be described on the diameter SA; as appears by Prop. 33. Then about the centre S, with an interval equal to half the latus rectum, describe the circle HkK; and at the place G of the ascending or descending body, and at any other place C, erect the perpendiculars GI, CD, meeting the conic section or circle in I and D. Then joining SI, SD, let the sectors HSK, HSk be made equal to the segments SEIS, SEDS, and (by Prop. 35.) the body G will describe the space GC in the same time in which the body K may describe the arc Kk. Q. E. F.

PROP. XXXVIII.

Supposing that the centripetal force is proportional to the altitude or distance of places from the centre, I say, that the times and velocities of falling bodies, and the spaces which they describe, are respectively proportional to the arcs, and the right and versed sines of the arcs. (Fig. 34.)

Suppose the body to fall from any point A in the right line AS; and about the centre of force S, with the interval AS, describe the quadrant of a circle AE; and let CD be the right sine of any arc AD; and the body A will in the time AD in falling describe the space AC, and in the place C will acquire the velocity CD.

This is demonstrated the same way from Prop. 10. as Prop. 32. was demonstrated from Prop. 11.

COR. 1. Hence the times are equal in which one body falling from the place A arrives at the centre S, and another body revolving describes the quadrantal arc ADE.

COR. 2. Wherefore all the times are equal in which bodies falling from whatsoever places arrive at the centre. For all the periodic times of revolving bodies are equal, (by Cor. 3, Prop. 4.)

PROP. XXXIX.

Supposing a centripetal force of any kind, and granting the quadratures of curvilinear figures; it is required to find the velocity of a body, ascending or descending in a right line, in the several places through which it passes; as also the time in which it will arrive at any place: and vice versâ.

Suppose the body E (Fig. 35.) to fall from any place A in the right line ADEC; and from its place E imagine a perpendicular EG always erected proportional to the centripetal force in that place tending to the centre C; and let BFG be a curve line, the locus of the point G. And in the beginning of the motion suppose EG to coincide with the perpendicular AB; and the velocity of the body in any place E will be as a right line whose power is the curvilinear area $\frac{1}{2}$ ABGE. Q. E. I.

In EG take EM reciprocally proportional to a right line whose power is the area $\frac{1}{2}$ ABGE, and let VLM be a curve line wherein the point M is always placed, and to which the right line AB produced is an asymptote; and the time in which the body falling describes the line AE, will be as the curvilinear area ABTVME. Q. E. I.

For in the right line AE let there be taken the very small line DE of a given length, and let DLF be the place of the line EMG, when the body was in D; and if the centripetal force be such, that a right line, whose power is the area $\frac{1}{2}$ ABGE, is as the velocity of the descending body, the area itself will be as the square of that velocity; that is, if for the velocities in D and E we write V and V+I, the area ABFD will be as VV, and the area ABGE as VV+2VI+II; and by division, the area DFGE as 2VI+II, and therefore $\frac{\text{DFGE}}{\text{DE}}$ will be as $\frac{2\text{VI}+\text{II}}{\text{DE}}$; that is, if we take the first ratios of those quantities when just nascent, the length DF is as the quantity $\frac{2\text{VI}}{\text{DE}}$, and therefore also as half that quantity $\frac{\text{I}\times\text{V}}{\text{DE}}$. But the time in which the body in falling describes the very small line DE, is as that line directly, and the velocity V inversely; and the force will be as the increment I of the velocity directly and the time inversely; and therefore if we take the first ratios when those quantities are just nascent, as $\frac{\text{I}\times\text{V}}{\text{DE}}$, that is, as the length DF. Therefore a force proportional to DF or EG will cause the body to descend with a velocity that is as the right line whose power is the area $\frac{1}{2}$ ABGE. Q. E. D.

Moreover, since the time in which a very small line DE of a given length may be described is as the velocity inversely, and therefore also inversely as a right line whose square is equal to the area ABFD; and since the line DL, and by consequence the nascent area DLME, will be as the same right line inversely, the time will be as the area DLME, and the sum of all the times will be as the sum of all the areas; that is (by Cor. Lem. 4.) the whole time in which the line AE is described will be as the whole area ATVME. Q. E. D.

Cor. 1. Let P be the place from whence a body ought to fall, so as that, when urged by any known uniform centripetal force, (such as gravity is vulgarly supposed to be), it may acquire in the place D a velocity equal to the velocity which another body, falling by any force whatever, hath acquired in that place D. In the perpendicular DF let there be taken DR, which may be to DF as that uniform force to the other force in the place D. Complete the rectangle PDRQ, and cut off the area ABFD equal to that rectangle. Then A will be the place from whence the other body fell. For completing the rectangle DRSE, since the area ABFD is to the area DFGE as VV to 2 VI, and therefore as $\frac{1}{2}$ V to I, that is, as half the whole velocity to the increment of the velocity of the body falling by the unequable force; and in like manner the area PQRD to the area DRSE as half the whole velocity to the increment of the velocity of the body falling by the uniform force; and since those increments (by reason of the equality of the nascent times) are as the generating forces, that is, as the ordinates DF, DR, and consequently as the nascent areas DFGE, DRSE; therefore, *ex æquo*, the whole areas ABFD, PQRD will be to one another as the halves of the whole velocities; and therefore, because the velocities are equal, they become equal also.

COR. 2. Whence if any body be projected either upwards or downwards with a given velocity from any place D, and there be given the law of centripetal force acting on it, its velocity will be found in any other place, as e, by erecting the ordinate eg, and taking that velocity to the velocity in the place D as a right line whose square is equal to the rectangle PQRD, either increased by the curvilinear area DFge, if the place e is below the place D, or diminished by the same area DFge, if it be higher, is to the right line whose square is equal to the rectangle PQRD alone.

COR. 3. The time is also known by erecting the ordinate em reciprocally proportional to the square root of PQRD + or — DFge, and taking the time in which the body has described the line De to the time in which another body has fallen with an uniform force from P, and in falling arrived at D in the proportion of the curvilinear area DLme to the rectangle 2 PD × DL. For the time in which a body falling with an uniform force hath described the line PD, is to the time in which the same body has described the line PE in the subduplicate ratio of PD to PE; that is (the very small line DE being just nascent), in the ratio of PD to $PD+\frac{1}{2}DE$, or 2 PD to $2PD+DE$, and, by division, to the time in which the body hath described the small line DE, as 2 PD to DE, and therefore as the rectangle 2 PD × DL to the area DLME; and the time in which both the bodies described the very small line DE is to the time in which the body moving unequably hath described the line De as the area DLME to the area DLme; and, *ex æquo*, the first mentioned of these times is to the last as the rectangle 2 PD × DL to the area DLme.

SECTION VIII.

PROP. XL.

If a body, acted upon by any centripetal force, be any how moved, and another body ascend or descend in a right line, and their velocities be equal in any one case of equal altitudes, their velocities will be also equal at all equal altitudes.

Let a body descend from A (Fig. 36.) through D and E, to the centre C; and let another body move from V in the curve line VIKk. From the centre C, with any distances, describe the concentric circles DI, EK, meeting the right line AC in D and E, and the curve VIK in I and K. Draw IC meeting KE in N, and on IK let fall the perpendicular NT; and let the interval DE or IN between the circumferences of the circles be very small; and imagine the bodies in D and I to have equal velocities. Then because the distances CD and CI are equal, the centripetal forces in D and I will be also equal. Let those forces be expressed by the equal *lineolæ* DE and IN; and let the force IN (by Cor. 2. of the Laws of Motion) be resolved into two others, NT and IT. Then the force NT acting in the direction of the line NT perpendicular to the path ITK of the body will not at all affect or change the velocity of the body in that path, but only draw it aside from a rectilinear course, and make it deflect perpetually from the tangent of the orbit, and proceed in the curvilinear path ITKk. That whole force, therefore, will be spent in producing this effect; but the other force IT, acting in the direction of the course of the body, will be all employed in accelerating it, and in the least given time will produce an acceleration proportional to itself. Therefore the accelerations of the bodies in D and I, produced in equal times, are as the lines DE, IT (if we take the first ratios of the nascent lines DE, IN, IK, IT, NT); and in unequal times as those lines and the times conjunctly. But the times in which DE and IK are described, are, by reason of the equal velocities (in D and I) as the spaces described DE and IK, and therefore the accelerations in the course of the bodies through the lines DE and IK are as DE and IT, and DE and IK conjunctly; that is, as the square of DE to the rectangle IT into IK. But the rectangle IT × IK is equal to the square of IN, that is, equal to the square of DE; and therefore the accelerations generated in the passage of the bodies from D and I to E and K are equal. Therefore the velocities of the bodies in E and K are also equal: and by the same reasoning they will always be found equal in any subsequent equal distances. Q. E. D.

By the same reasoning, bodies of equal velocities and equal distances from the centre will be equally retarded in their ascent to equal distances. Q. E. D.

Cor. 1. Therefore if a body either oscillate by hanging to a string, or by any polished and perfectly smooth impediment be forced to move in a curve line; and another body ascend or descend in a right line, and their velocities be equal at any one equal altitude, their velocities will be also equal at all other equal altitudes. For, by the string of the pendulous body, or by the impediment of a vessel perfectly smooth, the same thing will be effected as by the transverse force NT. The body is neither accelerated nor retarded by it, but only is obliged to quit its rectilinear course.

Cor. 2. Suppose the quantity P to be the greatest distance from the centre to which a body can ascend, whether it be oscillating, or revolving in a trajectory, and so the same projected upwards from any point of a trajectory with the velocity it has in that point. Let the quantity A be the distance of the body from the centre in any other point of the orbit; and let the centripetal force be always as the power A^{n-1} of the quantity A, the index of which power $n-1$ is any number n diminished by unity. Then the velocity in every altitude A will be as $\sqrt{P^n - A^n}$, and therefore will be given. For by Prop. 39. the velocity of a body ascending and descending in a right line is in that very ratio.

SECTION XI.

OF THE MOTIONS OF BODIES TENDING TO EACH OTHER WITH CENTRIPETAL FORCES.

I have hitherto been treating of the attractions of bodies towards an immoveable centre; though very probably there is no such thing existing in nature. For attractions are made towards bodies, and the actions of the bodies attracted and attracting are always reciprocal and equal, by Law 3; so that if there are two bodies, neither the attracted nor the attracting body is really at rest, but both (by Cor. 4. of the Laws of Motion) being as it were mutually attracted, revolve about a common centre of gravity. And if there be more bodies, which are either attracted by one single one which is attracted by them again, or which, all of them, attract each other mutually; these bodies will be so moved among themselves, as that their common centre of gravity will either be at rest, or move uniformly forward in a right line. I shall therefore at present go on to treat of the motion of two bodies mutually attracting each other; considering the centripetal forces as attractions; though perhaps in a physical strictness they may more truly be called impulses. But these propositions are to be considered as purely mathematical; and therefore laying aside all physical considerations, I make use of a familiar way of speaking, to make myself the more easily understood by a mathematical reader.

PROP. LVII.

Two bodies attracting each other describe similar figures about their common centre of gravity, and about each other.

For the distances of the bodies from their common centre of gravity are reciprocally as the bodies; and therefore in a given ratio to each other; and thence, by composition of ratios, in a given ratio to the whole distance between the bodies. Now these distances revolve about their common extremity with an equable angular motion, because lying in the same ri[illegible]line they never change their inclination to each oth[illegible]But right lines that are in a given ratio to each other, and revolve about extremities with an equal angular motion, describe upon planes, which either rest or move with any motion *not angular*, figures entirely similar around those extremities. Therefore the figures described by the revolution of these distances are similar. Q. E. D.

PROP. LVIII.

If two bodies attract each other with forces of any kind, and in the mean time revolve about their common centre of gravity; I say, that, by the same forces, there may be described round either body unmoved a figure similar and equal to the figures which the bodies so moving describe round each other.

Let the bodies S and P (Fig. 37.) revolve about their common centre of gravity C, proceeding from S to T, and from P to Q. From the given point s let there be continually drawn sp, sq, equal and parallel to SP, TQ; and the curve pqv, which the point p describes in its revolution round the immoveable point s, will be similar and equal to the curves which the bodies S and P describe about each other; and therefore, by Prop. 57., similar to the curves ST and PQV which the same bodies describe about their common centre of gravity C; and that because the proportions of the lines SC, CP, and SP or sp, to each other, are given.

Case 1. The common centre of gravity C (by Cor. 4. of the Laws) is either at rest, or moves uniformly in a right line. Let us first suppose it at rest, and in s and p let there be placed two bodies, one immoveable in s, the other moveable in p, similar and equal to the bodies S and P. Then let the right lines PR and pr touch the curves PQ and pq in P and p, and produce CQ and sq to R and r. And because the figures CPRQ, sprq are similar, RQ will be to rq as CP to sp, and therefore in a given ratio. Hence if the force with which the body P is attracted towards the body S, and consequently towards the intermediate point C, were to the force with which the body p is attracted towards the centre s, in the same given ratio, these forces would in equal times attract the bodies from the tangents PR, pr to the arcs PQ, pq, through the intervals proportional to them RQ, rq; and therefore this last force (tending to s) would make the body p revolve in the curve pqv, which would become similar to the curve PQV, in which the first force obliges the body P to revolve; and their revolutions would be completed in the same times. But because those forces are not to each other in the ratio of CP to sp, but (by reason of the similarity and equality of the bodies S and s, P and p, and the equality of the distances SP, sp) equal, the bodies in equal times will be equally drawn from the tangents; and therefore that the body p may be attracted through the greater interval rq, there is required a greater time, which will be in the subduplicate ratio of the intervals; because, by Lemma 10. the spaces described at the very beginning of the motion are in a duplicate ratio of the times. Suppose, then, the velocity of the body p to be to the velocity of the body P in a subduplicate ratio of the distance sp to the distance CP, so that the arcs pq, PQ, which are in a simple proportion to each other, may be described in times that are in a subduplicate ratio of the distances; and the bodies P, p, always attracted by equal forces, will describe round the quiescent centres C and s similar figures PQV, pqv, the latter of which pqv is similar and equal to the figure which the body P describes round the moveable body S. Q. E. D.

CASE 2. Suppose now that the common centre of gravity, together with the space in which the bodies are moved among themselves, proceeds uniformly in a right line; and (by Cor. 6. of the Laws), all the motions in this space will be performed in the [illegible] manner as before; and therefore the bodies will describe about each other the same figures as before, which will be therefore similar and equal to the figure pqv. Q. E. D.

COR. 1. Hence two bodies attracting each other with forces proportional to their distance, describe (by Prop. 10.) both round their common centre of gravity, and round each other, concentrical ellipses; and, *vice versâ*, if such figures be described, the forces are proportional to the distances.

COR. 2. And two bodies, whose forces are reciprocally proportional to the square of their distance, describe (by Prop. 11, 12, 13), both round their common centre of gravity, and round each other, conic sections having their focus in the centre about which the figures are described. And, *vice versâ*, if such figures be described, the centripetal forces are reciprocally proportional to the squares of the distance.

COR. 3. Any two bodies revolving round their common centre of gravity describe areas proportional to the times, by radii drawn both to that centre and to each other.

PROP. LIX.

The periodic time of two bodies S *and* P *revolving round their common centre of gravity* C, *is to the periodic time of one of the bodies* P *revolving round the other* S *remaining unmoved, and describing a figure similar and equal to those which the bodies describe about each other, in a subduplicate ratio of the other body* S *to the sum of the bodies* S + P.

For, by the demonstration of the last proposition, the times in which any similar arcs PQ and pq are described are in a subduplicate ratio of the distances CP and SP, or sp, that is, in a subduplicate ratio of the body S to the sum of the bodies S + P. And by composition of ratios, the sums of the times in which all the similar arcs PQ and pq are described, that is, the whole times in which the whole similar figures are described, are in the same subduplicate ratio. Q. E. D.

PROP. LX.

If two bodies S *and* P, *attracting each other with forces reciprocally proportional to the squares of their distance, revolve about their common centre of gravity; I say, that the principal axis of the ellipse which either of the bodies, as* P, *describes by this motion about the other* S, *will be to the principal axis of the ellipse, which the same body* P *may describe in the same periodic time about the other body* S *quiescent, as the sum of the two bodies* S+P *to the first of two mean proportionals between that sum and the other body* S.

For if the ellipses described were equal to each other, their periodic times by the last theorem would be in a subduplicate ratio of the body S to the sum of the bodies S+P. Let the periodic time in the latter ellipse be diminished in that ratio, and the periodic times will become equal; but, by Prop. 15. the principal axis of the ellipse will be diminished in a ratio sesquiplicate of the former ratio; that is, in a ratio to which the ratio of S to S+P is triplicate; and therefore that axis will be to the principal axis of the other ellipse as the first of two mean proportionals between S+P and S to S+P. And inversely the principal axis of the ellipse described about the moveable body will be to the principal axis of that described round the immoveable as S+P to the first of two mean proportionals between S+P and S. Q. E. D.

PROP. LXI.

If two bodies attracting each other with any kind of forces, and not otherwise agitated or obstructed, be moved in any manner whatsoever, those motions will be the same as if *they did not at all attract each other, but were both attracted with the same forces by a third body placed in their common centre of gravity; and the law of the attracting forces will be the same in respect of the distance of the bodies from the common centre, as in respect of the distance between the two bodies.*

For those forces with which the bodies attract each other, by tending to the bodies, tend also to the common centre of gravity lying directly between them; and therefore are the same as if they proceeded from an intermediate body. Q. E. D.

And because there is given the ratio of the distance of either body from that common centre to the distance between the two bodies, there is given, of course, the ratio of any power of one distance to the same power of the other distance; and also the ratio of any quantity derived in any manner from one of the distances compounded any how with given quantities, to another quantity derived in like manner from the other distance, and as many given quantities having that given ratio of the distances to the first. Therefore if the force with which one body is attracted by another be directly or inversely as the distance of the bodies from each other, or as any power of that distance; or, lastly, as any quantity derived after any manner from that distance compounded with given quantities; then will the same force with which the same body is attracted to the common centre of gravity be in like manner directly or inversely as the distance of the attracted body from the common centre, or as any power of that distance; or, lastly, as a quantity derived in like sort from that distance compounded with analogous given quantities. That is, the law of attracting force will be the same with respect to both distances. Q. E. D.

PROP. LXII.

To determine the motions of two bodies which attract each other with forces reciprocally proportional to the squares of the distance between them, and are let fall from given places.

The bodies, by the last theorem, will be moved in the same manner as if they were attracted by a third placed in their common centre of gravity; and by hypothesis, that centre will be quiescent at the beginning of the motion, and therefore (by Cor. 4. of the Laws of Motion) will be always quiescent. The motions of the bodies are therefore to be determined (by Prob. 25.) in the same manner as if they were impelled by forces tending to that centre; and then we shall have the motions of the bodies attracting each other. Q. E. I.

PROP. LXIII.

To determine the motions of two bodies attracting each other with forces reciprocally proportional to the squares of their distance, and going off from given places in given directions with given velocities.

The motions of the bodies at the beginning being given, there is given also the uniform motion of the common centre of gravity, and the motion of the space which moves along with this centre uniformly in a right line, and also the very first motions of the bodies in respect of this space. Then (by Cor. 5. of the Laws, and the last Theorem) the subsequent motions will be performed in the same manner in that space, as if that space together with the common centre of gravity were at rest, and as if the bodies did not attract each other, but were attracted by a third body placed in that centre. The motion therefore in this moveable space of each body going off from a given place, in a given direction, with a given velocity, and acted upon by a centripetal force tending to that centre, is to be determined by Prop. 17. and 37., and at the same time will be obtained the motion of the other round the same centre. With this motion compound the uniform progressive motion of the entire system of the space and the bodies revolving in it, and there will be obtained the absolute motion of the bodies in immoveable space. Q. E. I.

SECTION XII.

OF THE ATTRACTIVE FORCES OF SPHERICAL BODIES.

PROP. LXX.

If to every point of a spherical surface there tend equal centripetal forces decreasing in the duplicate ratio of the distances from those points; I say, that a corpuscle placed within that superficies will not be attracted by those forces any way.

Let HIKL (Fig. 38.) be that spherical superficies, and P a corpuscle placed within. Through P let there be drawn to this superficies the two lines HK, IL, intercepting very small arcs HI, KL; and because (by Cor. 3. Lem. 7.) the triangles HPI, LPK are similar, those arcs will be proportional to the distances HP, LP; and any particles at HI and KL of the spherical superficies, terminated by right lines passing through P, will be in the duplicate ratio of those distances. Therefore the forces of these particles exerted upon the body P are equal.

For the forces are as the particles directly, and the squares of the distances inversely. And these two ratios compose the ratio of equality. The attractions, therefore, being made equally towards contrary parts, destroy each other. And by a like reasoning all the attractions through the whole spherical superficies are destroyed by contrary attractions. Therefore the body P will not be any way impelled by those attractions. Q. E. D.

PROP. LXXI.

The same things supposed as above, I say, that a corpuscle placed without the spherical superficies is attracted towards the centre of the sphere with a force reciprocally proportional to the square of its distance from that centre.

Let AHKB, ahkb (Fig. 39.) be two equal spherical superficies described about the centres S, s,; their diameters AB, ab; and let P and p be two corpuscles situate without the spheres in those diameters produced. Let there be drawn from the corpuscles the lines PHK, PIL, phk, pil, cutting off from the great circles AHB, ahb, the equal arcs HK, hk, IL, il; and to those lines let fall the perpendiculars SD, sd, SE, se, IR, ir; of which let SD, sd, cut PL, pl, in F and f. Let fall also to the diameters the perpendiculars IQ, iq. Let now the angles DPE, dpe vanish; and because DS and ds, ES and es are equal, the lines PE, PF, and pe, pf, and the *lineolæ* DF, df may be taken for equal; because their last ratio, when the angles DPE, dpe vanish together, is the ratio of equality. These things then supposed, it will be, as PI to PF so is RI to DF, and as pf to pi so is df or DF to ri; and, *ex æquo*, as PI × pf to PF × pi so is RI to ri, that is (by Cor. 3. Lem. 7.) so is the arc IH to the arc ih. Again, PI is to PS as IQ to SE, and ps to pi as se or SE to iq; and, *ex æquo*, PI × ps to PS × pi as IQ to iq. And compounding the ratios $PI^2 \times pf \times ps$ is to $pi^2 \times PF \times PS$, as

$IH \times IQ$ to $ih \times iq$; that is, as the circular superficies which is described by the arc IH, as the semi-circle AKB revolves about the diameter AB, is to the circular superficies described by the arc ih as the semi-circle akb revolves about the diameter ab. And the forces with which these superficies attract the corpuscles P and p in the direction of lines tending to those superficies are by hypothesis as the superficies themselves directly, and the squares of the distances of the superficies from those corpuscles inversely; that is, as $pf \times ps$ to $PF \times PS$. And these forces again are to the oblique parts of them which (by the resolution of forces as in Cor. 2. of the Laws) tend to the centres in the directions of the lines PS, ps, as PI to PQ, and pi to pq; that is (because of the like triangles PIQ and PSF, piq and psf), as PS to PF and ps to pf. Thence, *ex æquo*, the attraction of the corpuscle P towards S is to the attraction of the corpuscle p towards s as $\frac{PF \times pf \times ps}{PS}$ is to $\frac{pf \times PF \times PS}{ps}$, that is, as ps^2 to PS^2. And, by a like reasoning, the forces with which the superficies described by the revolution of the arcs KL, kl attract those corpuscles, will be as ps^2 to PS^2. And in the same ratio will be the forces of all the circular superficies into which each of the spherical superficies may be divided by taking sd always equal to SD, and se equal to SE. And therefore, by composition, the forces of the entire spherical superficies exerted upon those corpuscles will be in the same ratio. Q. E. D.

PROP. LXXII.

If to the several points of a sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from those points; and there be given both the density of the sphere and the ratio of the diameter of the sphere to the distance of the corpuscle from its centre; I say, that the corpuscle from the centre of the sphere (by Prop. 71.) And, by composition, the sum of those attractions, that is, the attraction of the corpuscle towards the entire sphere, will be in the same ratio. Q. E. D.

COR. 1. Hence the attractions of homogeneous spheres at equal distances from the centres will be as the spheres themselves. For (by Prop. 72.) if the distances be proportional to the diameters of the spheres, the forces will be as the diameters. Let the greater distance be diminished in that ratio; and the distances now being equal, the attraction will be increased in the duplicate of that ratio; and therefore will be to the other attraction in the triplicate of that ratio of the spheres.

COR. 2. At any distances whatever the attractions are as the spheres applied to the squares of the distances.

COR. 3. If a corpuscle placed without an homogeneous sphere is attracted by a force reciprocally proportional to the square of its distance from the centre, and the sphere consist of attractive particles, the force of every particle will decrease in a duplicate ratio of the distance from each particle.

PROP. LXXV.

If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that another similar sphere will be attracted by it with a force reciprocally proportional to the square of the distance of the centres.

For the attraction of every particle is reciprocally as the square of its distance from the centre of the attracting sphere, (by Prop. 74.), and is therefore the same as if that whole attracting force issued from one single corpuscle placed in the centre of this sphere. But this attraction is as great as on the other hand the attraction of the same corpuscle would be, if that were itself attracted by the *tances from the points; I say, that a corpuscle placed within the sphere is attracted by a force proportional to its distance from the centre.*

In the sphere ABCD (Fig. 40.) described about the centre S, let there be placed the corpuscle P; and about the same centre S, with the interval SP, conceive described an interior sphere PEQF. It is plain (by Prop. 70.) that the concentric spherical superficies, of which the difference AEBF of the spheres is composed, have no effect at all upon the body P, their attractions being destroyed by contrary attractions. There remains, therefore, only the attraction of the interior sphere PEQF. And (by Prop. 72.) this is as the distance PS. Q. E. D.

SCHOLIUM.

By the superficies of which I here imagine the solids composed, I do not mean superficies purely mathematical, but orbs so extremely thin, that their thickness is as nothing; that is, the evanescent orbs of which the sphere will at last consist, when the number of the orbs is increased, and their thickness diminished without end. In like manner, by the points of which lines, surfaces, and solids are said to be composed, are to be understood equal particles, whose magnitude is perfectly inconsiderable.

PROP. LXXIV.

The same things supposed, I say, that a corpuscle, situate without the sphere is attracted with a force reciprocally proportional to the square of its distance from the centre.

For suppose the sphere to be divided into innumerable concentric spherical superficies, and the attractions of the corpuscle arising from the several superficies will be reciprocally proportional to the square of the distance of the corpuscle from the centre of the sphere (by Prop. 71.) And, by composition, the sum of those attractions, that is, the attraction of the corpuscle towards the entire sphere, will be in the same ratio. Q. E. D.

COR. 1. Hence the attractions of homogeneous spheres at equal distances from the centres will be as the spheres themselves. For (by Prop. 72.) if the distances be proportional to the diameters of the spheres, the forces will be as the diameters. Let the greater distance be diminished in that ratio; and the distances now being equal, the attraction will be increased in the duplicate of that ratio; and therefore will be to the other attraction in the triplicate of that ratio of the spheres.

COR. 2. At any distances whatever the attractions are as the spheres applied to the squares of the distances.

COR. 3. If a corpuscle placed without an homogeneous sphere is attracted by a force reciprocally proportional to the square of its distance from the centre, and the sphere consist of attractive particles, the force of every particle will decrease in a duplicate ratio of the distance from each particle.

PROP. LXXV.

If to the several points of a given sphere there tend equal centripetal forces decreasing in a duplicate ratio of the distances from the points; I say, that another similar sphere will be attracted by it with a force reciprocally proportional to the square of the distance of the centres.

For the attraction of every particle is reciprocally as the square of its distance from the centre of the attracting sphere, (by Prop. 74.), and is therefore the same as if that whole attracting force issued from one single corpuscle placed in the centre of this sphere. But this attraction is as great as on the other hand the attraction of the same corpuscle would be, if that were itself attracted by the several particles of the attracted sphere with the same force with which they are attracted by it. But that attraction of the corpuscle would be (by Prop. 74.) reciprocally proportional to the square of its distance from the centre of the sphere; therefore the attraction of the sphere, equal thereto, is also in the same ratio. Q. E. D.

COR. 1. The attractions of spheres towards other homogeneous spheres are as the attracting spheres applied to the squares of the distances of their centres from the centres of those which they attract.

COR. 2. The case is the same when the attracted sphere does also attract. For the several points of the one attract the several points of the other with the same force with which they themselves are attracted by the others again; and therefore since in all attractions (by Law 3.) the attracted and attracting point are both equally acted on, the force will be doubled by their mutual attractions, the proportions remaining.

COR. 3. Those several truths demonstrated above, concerning the motion of bodies about the focus of the conic sections, will take place when an attracting sphere is placed in the focus, and the bodies move without the sphere.

COR. 4. Those things which were demonstrated before of the motion of bodies about the centre of the conic sections take place when the motions are performed within the sphere.

PROP. LXXVI.

If spheres be however dissimilar (as to density of matter and attractive force) in the same ratio onward from the centre to the circumference; but every where similar, at every given distance from the centre, on all sides round about; and the attractive force of every point decreases in the duplicate ratio of the distance of the body attracted; I say, that the whole force with which one of these spheres attracts *the other will be reciprocally proportional to the square of the distance of the centres.*

Imagine several concentric similar spheres, AB, CD, EF, &c. (Fig. 41.) the innermost of which added to the outermost may compose a matter more dense towards the centre, or subducted from them may leave the same more lax and rare. Then, by Prop. 75. these spheres will attract other similar concentric spheres GH, IK, LM, &c. each the other, with forces reciprocally proportional to the square of the distance SP. And, by composition or division, the sum of all those forces, or the excess of any of them above the others; that is, the entire force with which the whole sphere AB (composed of any concentric spheres or of their differences) will attract the whole sphere GH (composed of any concentric spheres or their differences) in the same ratio. Let the number of the concentric spheres be increased *in infinitum*, so that the density of the matter together with the attractive force may, in the progress from the circumference to the centre, increase or decrease according to any given law; and by the addition of matter not attractive, let the deficient density be supplied, that so the spheres may acquire any form desired; and the force with which one of these attracts the other will be still, by the former reasoning, in the same ratio of the square of the distance inversely. Q. E. D.

COR. 1. Hence if many spheres of this kind, similar in all respects, attract each other mutually, the accelerative attractions of each to each, at any equal distances of the centres, will be as the attracting spheres.

COR. 2. And at any unequal distances, as the attracting spheres applied to the squares of the distances between the centres.

COR. 3. The motive attractions, or the weights of the spheres towards one another, will be at equal distances of the centres as the attracting and attracted spheres conjunctly; that is, as the products arising from multiplying the spheres into each other.

Cor. 4. And at unequal distances, as those products directly, and the squares of the distances between the centres inversely.

Cor. 5. These proportions take place also when the attraction arises from the attractive virtue of both spheres mutually exerted upon each other. For the attraction is only doubled by the conjunction of the forces, the proportions remaining as before.

Cor. 6. If spheres of this kind revolve about others at rest, each about each; and the distances between the centres of the quiescent and revolving bodies be proportional to the diameters of the quiescent bodies; the periodic times will be equal.

Cor. 7. And, again, if the periodic times be equal, the distances will be proportional to the diameters.

Cor. 8. All those truths above demonstrated, relating to the motions of bodies about the foci of conic sections, will take place when an attracting sphere, of any form and condition like that above described, is placed in the focus.

Cor. 9. And also when the revolving bodies are also attracting spheres of any condition like that above described.

PROP. LXXVII.

If to the several points of spheres there tend centripetal forces proportional to the distances of the points from the attracted bodies; I say, that the compounded force with which two spheres attract each other mutually is as the distance between the centres of the spheres.

Case 1. Let AEBF (Fig. 42.) be a sphere; S its centre; P a corpuscle attracted; PASB the axis of the sphere passing through the centre of the corpuscle; EF, ef two planes cutting the sphere, and perpendicular to the axis, and equidistant, one on one side, the other on the other, from the centre of the sphere; G and g the intersections of the planes and the axis; and H any point in the plane EF. The centripetal force of the point H upon the corpuscle P, exerted in the direction of the line PH, is as the distance PH; and (by Cor. 2. of the Laws) the same exerted in the direction of the line PG, or towards the centre S, is as the length PG. Therefore the force of all the points in the plane EF (that is, of that whole plane) by which the corpuscle P is attracted towards the centre S is as the distance PG multiplied by the number of those points, that is, as the solid contained under that plane EF and the distance PG. And in like manner the force of the plane ef, by which the corpuscle P is attracted towards the centre S, is as that plane drawn into its distance Pg, or as the equal plane EF drawn into that distance Pg; and the sum of the forces of both planes as the plane EF drawn into the sum of the distances PG + Pg, that is, as that plane drawn into twice the distance PS of the centre and the corpuscle; that is, as twice the plane EF drawn into the distance PS, or as the sum of the equal planes EF + ef drawn into the same distance. And, by a like reasoning, the forces of all the planes in the whole sphere, equi-distant on each side from the centre of the sphere, are as the sum of those planes drawn into the distance PS, that is, as the whole sphere and the distance PS conjunctly. Q. E. D.

CASE 2. Let now the corpuscle P attract the sphere AEBF. And, by the same reasoning, it will appear that the force with which the sphere is attracted is as the distance PS. Q. E. D.

CASE 3. Imagine another sphere composed of innumerable corpuscles P; and because the force with which every corpuscle is attracted is as the distance of the corpuscle from the centre of the first sphere, and as the same sphere conjunctly, and is therefore the same as if it all proceeded from a single corpuscle situate in the centre of the sphere, the entire force with which all the corpuscles in the second sphere are attracted, that is, with which that whole sphere is attracted, will be the same as if that sphere were attracted by a force issuing from a single corpuscle in the centre of the first sphere; and is therefore proportional to the distance between the centres of the spheres. Q. E. D.

Case 4. Let the spheres attract each other mutually, and the force will be doubled, but the proportion will remain. Q. E. D.

Case 5. Let the corpuscle p be placed within the sphere AEBF (Fig. 43.); and because the force of the plane ef upon the corpuscle is as the solid contained under that plane and the distance pg; and the contrary force of the plane EF as the solid contained under that plane and the distance pG; the force compounded of both will be as the difference of the solids, that is, as the sum of the equal planes drawn into half the difference of the distances; that is, as that sum drawn into pS, the distance of the corpuscle from the centre of the sphere. And, by a like reasoning, the attraction of all the planes EF, ef, throughout the whole sphere, that is, the attraction of the whole sphere, is conjointly as the sum of all the planes, or as the whole sphere, and as pS, the distance of the corpuscle from the centre of the sphere. Q. E. D.

Case 6. And if there be composed a new sphere out of innumerable corpuscles such as p, situate within the first sphere AEBF, it may be proved, as before, that the attraction, whether single of one sphere towards the other, or mutual of both towards each other, will be as the distance pS of the centres. Q. E. D.

PROP. LXXVIII.

If spheres in the progress from the centre to the circumference be however dissimilar and unequable, but similar on every side round about at all given distances from the centre; and the attractive force of every point be as the distance of the attracted body; I say, that the entire force with which two spheres of this kind attract each other mutually is proportional to the distance between the centres of the spheres.

This is demonstrated from the foregoing proposition, in the same manner as the 76th proposition was demonstrated from the 75th.

Cor. Those things that were above demonstrated in Prop. 10 and 64. of the motion of bodies round the centres of conic sections, take place when all the attractions are made by the force of spherical bodies of the condition above described, and the attracted bodies are spheres of the same kind.

AN

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Definitions and Principles.

(1.) The nature of that singular modification in virtue of which a body is transferred from one place to another is and will ever be unknown: it has been designated by the name of *force*; we can only determine its effects and the laws of its action.

(2.) There are three things to be determined of a force statically considered, viz. its point of application, its intensity, and its direction. The first is determined by referring the body to three rectangular axes of coordinates arbitrarily adopted; the second by adopting some unit of force, and considering every other as bearing to it some numerical ratio; and the third by determining the angles made by the direction of the force with the three assumed axes.

(3.) When forces are *dynamically* considered, it moreover becomes necessary to determine two additional circumstances:—1st, whether forces are the only causes that can produce or prevent motion in a body; 2d, whether the velocity which results from the action of a force bears any assignable relation to the force producing it, and what this relation is.

(4.) As to the first of these circumstances, it is only necessary to shew that a body cannot possibly produce or prevent motion in itself, for there are no other causes supposed to produce or prevent motion except the force applied and the moving body. That a body cannot produce motion in itself is a circumstance which has been always allowed true, but it is not so immediately obvious that a body may not stop of itself, particularly as the gradual cessation of every motion produced by human exertion tends strongly to confirm this opinion. The cessation of terrestrial motions can, however, be always traced to the operation of some external hindrances, and is by no means to be ascribed to any tendency the moving body has to rest rather than to motion. Experiment proves that the more we diminish the resistance of air, friction, &c. the longer terrestrial motions continue; and thence we conclude by analogy, that if these resistances could be altogether removed, the motion would continue undiminished for ever. This analogy is abundantly confirmed by the motions of the bodies of the planetary system, since they have persevered in their motions about their respective primaries without any sensible retardation, though unsupported in this motion by any *material* agency.

(5.) When we consider forces as producing equilibrium, it is only necessary to compare them by their relative powers of counteraction. In Dynamics, however, it becomes necessary to compare forces with their effects, or, in other words, to determine what relation the force impressed on any material point and the consequent velocity bear to each other. As force is only known by the space it makes a body pass over in a determinate time, it is natural to take this space as its measure, but this supposes that several forces acting in the same direction will cause a body to pass over a space equal to the sum of the spaces which each of them separately would cause it to pass over, or, which is the same thing, that the force is proportional to the velocity. This is a circumstance which we cannot possibly know *a priori* from our ignorance of the nature of motive force. We must then on this subject have recourse to experiment, for every thing not mathematically deducible from the few *data* that we have on the nature of things, is with us only a result of observation.

(6.) If we conceive a globe at rest upon a perfectly smooth horizontal plane, to be struck by a right angled cylinder moving in the direction of its axis supposed horizontal; the apparent relative motion of the globe will always be found coincident with the direction of the force impressed. The most accurate experiments have not indicated any the slightest deviation from this direction. That this circumstance infers the proportionality of the force to the velocity, shall appear by the following analysis.

Let V be the velocity due to the motion of the earth, and which is common to all the bodies at its surface; let F be the force in consequence. The velocity being then some function of the force, which we shall denote by the character ϕ, we shall have in general $V=\phi.(F)$. Decomposing this force into three rectangular components a, b, c; and representing by x, y, z the components of the velocity along their directions respectively, we shall have $x=\phi(a)$, $y=\phi(b)$, $z=\phi(c)$.

Let us now suppose the moving body solicited by a force, F′, exclusive of that common to the whole earth, and let its components along the axes be a', b', c'; the whole forces along the axes will then be $(a+a')$, $(b+b')$, $(c+c')$; and the components of the total velocities along these same axes will be $\phi.(a+a')$, $\phi.(b+b')$, $\phi.(c+c')$. Designating by x' the increment of the velocity x due to the force F′, we have

$$x+x'=\phi.(a+a')=\phi.(a)+\frac{d.\phi.(a)}{da}.\frac{a'}{1}+\frac{d^2\phi.(a)}{da^2}.\frac{a'^2}{1.2}+\&c.$$

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$$x+x'=\phi.(a+a')=\phi.(a)+\frac{d.\phi.(a)}{da}.\frac{a'}{1}+\frac{d^2\phi.(a)}{da^2}.\frac{a'^2}{1.2}+\&c.$$

but $x=\phi.(a)$, therefore

$$x'=\frac{d.\phi.(a)}{da}.\frac{a'}{1}+\frac{d^2.\phi.(a)}{da^2}.\frac{a'^2}{1.2}+\&c.$$

Experiment proves that x' is independent of a, which requires that the second member of this equation should be independent of a. Hence, the co-efficients of the powers of a' should be independant of a, or be constant. We have then $\frac{d.\phi.(a)}{da}=k$, k being a constant. From this we deduce $\frac{d^2.\phi.(a)}{da^2}=0$, &c. and hence $x'=ka'$; and similarly $y'=kb'$, and $z'=kc'$; and, in general, we can from this conclude, that *if a force, f, impart a velocity, v, to a moving body, we shall always have $v=kf$, k being a constant quantity.*

This very elegant demonstration is taken from the *Annales des Mathem. tom.* xv. It is a simplification by M. Querret, of St. Malo, of a demonstration by M. Fauquier, given in the same work, *tom.* xiv.

(7.) These two laws, that of inertia and the proportionality of the force to the velocity, are results of observation, and are the foundation on which the whole of Dynamics is built. The science of Statics is more abstract in this respect, as it is founded on no hypothesis whatsoever.

(8.) From the proportionality of the force to the velocity it follows, that the relative motions of the bodies of a system are the same whatever be their common motion, for the common motion decomposed along three axes does nothing more than increase equally the partial velocities of each body parallel to the same axes; but the relative velocities merely depend upon the difference of these partial velocities, which difference is not altered by the common motion. It is, therefore, impossible to judge of the absolute motion of a system, of which we compose a part, by appearances observed, and this is the circumstance which characterizes the law of the proportionality of the force to the velocity.

(9.) Forces estimated by their effects may be divided into two classes:—1st, those which produce constant velocities; 2d, those which produce variable velocities: the former forces are single impulses, and impart their whole velocities in an instant; the latter are forces the action of which is continued, the velocity in this case depends upon the time during which the force acts, and is, therefore, nothing at the commencement of the motion.

(10.) The full theory of the motion of a point acted on by a single impulse is contained in the equation $s=vt+b$, where s, v, t are the space, time, and velocity; and b the value of s at the commencement of the motion. If any other point move along the same line, we have $s'=v't'+b'$; and if the points come together, we shall have $t=\frac{b-b'}{v'-v}$.

(11.) With respect to forces that produce varied velocities, the simplest are those which always continue the same. Of this description is the force of gravity acting on a material point *in vacuo* at the earth's surface. The velocity resulting from such a force is evidently uniformly varied, since the several successive impulses are equal and all in the same direction. Let t be the whole time of the motion, which we shall conceive divided into an indefinite number of instants; let g be the velocity at the end of the first instant, then $2g$, $3g$, &c. will be the velocities at the ends of the succeeding instants. If we conceive the velocity at the end of each instant to have acted during that instant, the whole space described shall be the series $g+2g+3g \ldots . tg$; the sum of this series gives us $S'=(g+tg).\frac{t}{2}$. The space with the velocities at the commencement of each instant, would be $0+g+2g \ldots (t-1).g$; the sum of which gives us $S''=(-g+gt).\frac{t}{2}$. If we consider the velocity at the beginning and end of each instant the same, we have $S' = S'' = S$; and hence, adding the values of S' and S'', we find $S = g. \frac{t^2}{2}$.

Let v be the final velocity, then from the nature of the motion we have $v = g. t$. Substituting for t from this in the value of s, we obtain $v = \sqrt{2gs}$.

The three equations,

$$v = g. t; \qquad s = \frac{g. t^2}{2}; \qquad v = \sqrt{2gs}. \qquad (1)$$

contain the whole theory of uniformly varied motion. This comprises the falling of bodies *in vacuo*, and the descent of bodies down inclined planes.

(12.) With respect to varied rectilinear motion in general, when the successively imparted velocities are not necessarily equal, we must have recourse to the theory of limits to determine the equations which subsist between the spaces, times, velocities, and forces. Let the whole time be divided into a number, n, of equal intervals; let θ be the length of one of these intervals; and let the velocities imparted in the successive instants be v, v', v'', &c.; then the whole space passed over in the time t, $(= n\theta)$, is $v\theta + (v+v')\theta + (v+v'+v'')\theta + \&c. = vn\theta + v'.(n-1)\theta + v''.(n-2)\theta + \&c.$; and the velocity of the moving point at the end of the time is $v+v'+v'' \ldots . v^{n-1}$.

The space passed over with this final velocity in the same time $= n\theta. (v+v'+v'' \ldots . v^{n-1})$, which is greater than the former space if v, v', v'' be all positive, or if the motion be accelerated.

Let $\phi(t) = s$ at the end of any time t, then $\phi.(t+t') - \phi(t)$ is equal to the space passed over in the time t'; let v be the velocity at the end of the time t, and $(v+u)$ at the end of the time $(t+t')$, then $\phi.(t+t') - \phi(t) < (v+u).t'$, and $> vt'$; hence, $\phi.(t+t') - \phi(t)$ is always comprised between these limits, and therefore, ultimately, we have

$\frac{\phi.(t+t')-\phi.(t)}{t'}=v$; or, using the ordinary notation for limiting ratios, we have

$$\frac{ds}{dt}=v. \tag{2}$$

If the force be a retarding one, $\phi(t+t')-\phi(t)<vt$ and $>(v-u)t$, therefore, &c., as before.

If a force, f, communicate to a moving point the velocity, g, in the unit of time, then the velocity imparted in the time dt is $g dt$, in which time suppose the force ϕ to impart the velocity dv, then $\phi : f :: dv : g dt$, hence, $\phi=\frac{f}{g}.\frac{dv}{dt}$. If f be the unit of force which produces g, the unit of velocity in the unit of time, then

$$\phi=\frac{dv}{dt}. \tag{3}$$

This value of ϕ would be an abstract number expressing its ratio to the unit of force.

Substituting for dv from the equation $v=\frac{ds}{dt}$, we find

$$\phi=\frac{d^2s}{dt^2}. \tag{4}$$

Another differential equation may be formed from the equations $v=\frac{ds}{dt}$, and $\phi=\frac{dv}{dt}$, by eliminating dt. Hence we have

$$d.v^2=2\phi.ds. \tag{5}$$

The four equations thus established are sufficient to determine all the circumstances of the motion of a body in a right line. We shall now proceed to establish the principles of curvilinear motion.

(13.) If a material point in free space be acted on by two forces of the species mentioned in Art. (10.), it will move uniformly over the diagonal of a parallelogram having the component forces for its sides. If in the course of its motion a third force be conceived applied not coinciding with the direction of the bodies motion, its direction and velocity will be altered. The motion will, however, be still rectilinear, and the velocity uniform. Conceive new forces thus to be successively applied, and the material point will, in consequence, describe a polygon which can be thus easily constructed. Conceive the forces to succeed each other at indefinitely small intervals, the motion will become continually varied, and the polygon shall become a curve. Thus we see how the transition might be made from rectilinear to curvilinear motion.

(14.) The simplest method of determining the motion of a material point in space is to assign at each instant the position of its projections on three fixed axes. We may represent these projections as three moving points along the fixed axes, and the question is then reduced to determine the laws of these three rectilinear motions.

Let R be the force acting on the moving point at any instant, and A, B, C the components of R parallel to the three assumed rectangular axes. Then by the parallelopiped of forces

$$A=R\cos.\alpha,\ B=R\cos.\beta,\ C=R\cos.\gamma,\ R^2=A^2+B^2+C^2.$$

Let the velocity due to R be v, and those to A, B, C; a, b, c respectively; then by Art. (6.), we have

$$a=v.\cos.\alpha,\quad b=v.\cos.\beta,\quad c=v.\cos.\gamma,\quad v^2=a^2+b^2+c^2.$$

The space described in the interval of time θ, in the direction of the motion, is $v\theta$, and the spaces along the axes are $v\theta\cos.\alpha$, $v\theta\cos.\beta$, $v\theta\cos.\gamma$, respectively; or, $a\theta$, $b\theta$, $c\theta$. Let a new force, A′, act, and it will produce velocities, a', b', c', which shall cause in the interval, θ', succeeding θ, spaces to be described along the axes $(a+a')\theta'$, $(b+b')\theta'$, $(c+c')\theta'$; and so on continually. Hence, at the end of any number of intervals the values of x, y, z may be expressed as follows:

$$x=x'+a\theta+(a+a')\theta' \ldots . (a+a'+a'' \ldots a^{(n)})\,\theta^{(n)}.$$

$$y=y'+b\theta+(b+b')\theta' \ldots . (b+b'+b'' \ldots b^{(n)})\,\theta^{(n)}.$$

$$z=z'+c\theta+(c+c')\theta' \ldots . (c+c'+c'' \ldots c^{(n)})\,\theta^{(n)}.$$

By means of these values we can assign the position of the moving point in space at any time.

(15.) These values of x, y, z depend solely upon the velocities produced by the forces respectively parallel to them, and, therefore, the motion along each axis is independant of the components parallel to the others. Hence we may conclude, that *if we decompose into three forces parallel to three rectangular axes, the forces which produce any curvilinear motion, and if we consider as moving points the projections on the axes, the motion along each axis shall be due to the forces parallel to this axis, and shall be the same as if the other forces were nothing.*

Thus, the solution of any problem relative to curvilinear motion is reduced to the consideration of three rectilinear motions, or but two when we can be certain by the nature of the question that the moving point ought to continue in the same plane.

(16.) The preceding Theorem is *essentially* founded on the proportionality of the force to the velocity, as otherwise it would not be true. Decomposing the whole force acting on the moving point into three, X, Y, Z parallel respectively to the axes of x, y, and z, we have by equation (4), and by what has been proved in Art. (14.), the three following equations:

$$\frac{d^2x}{dt^2}=\mathrm{X}, \qquad \frac{d^2y}{dt^2}=\mathrm{Y}, \qquad \frac{d^2z}{dt^2}=\mathrm{Z}. \qquad (6)$$

(17.) We shall apply these to the case in which the forces parallel to the axes are respectively nothing; we have in this case

$$\frac{d^2x}{dt^2}=0, \qquad \frac{d^2y}{dt^2}=0, \qquad \frac{d^2z}{dt^2}=0.$$

Integrating these equations, we find

$$x=at+a', \qquad y=bt+b', \qquad z=ct+c'.$$

The constants a, b, c in these integrals are the components of the velocity parallel to the axes; and a', b', c' are the coordinates of the point of departure. Taking the origin at the point of departure, and eliminating t, we find

$$x=\frac{a}{c}\cdot z, \qquad y=\frac{b}{c}\cdot z.$$

Hence the moving point describes a right line, with a velocity equal to $\sqrt{a^2+b^2+c^2}$.

This application comprises the problem solved in Art. (10.)

(18.) If the accelerating forces which act upon the moving point in its curvilinear motion, should suddenly cease to act, the moving point would go off in the direction of the tangent with an uniform velocity. The point will evidently, from its *inertia*, move uniformly with the velocity which it had at the moment the forces became suspended. Let v be this velocity; its components parallel to the axes are also uniform; these components are $\frac{dx}{dt}$, $\frac{dy}{dt}$, and $\frac{dz}{dt}$. Hence

$$v.\cos.\alpha=\frac{dx}{dt}, \qquad v.\cos.\beta=\frac{dy}{dt}, \qquad v.\cos.\gamma=\frac{dz}{dt};$$

but

$$v=\frac{ds}{dt},$$

therefore,

$$\cos. \alpha = \frac{dx}{ds}, \qquad \cos. \beta = \frac{dy}{ds}, \qquad \cos. \gamma = \frac{dz}{ds}.$$

But the cosines of the angles made by the tangent with the axes are $\frac{dx}{ds}$, $\frac{dy}{ds}$, and $\frac{dz}{ds}$. Hence the direction of the motion coincides with the tangent.

SECTION II.

ON THE RECTILINEAR MOTION OF A MATERIAL POINT IN FREE SPACE.

(19.) The simplest application that can be given of the equations which have been proved in the preceding chapter, is to determine the motion of a material point along a right line. We shall suppose the point acted on by a single fixed centre of force, situated in the line along which it is supposed to move.

PROP. I.

(20.) *To determine a general expression for the velocity in moving along a right line by the action of a single centre.*

Let r be the distance of the moving point from the fixed centre at any instant of the motion; let the absolute force of the centre, or force at the unit of distance, be μ; and first let the force be inversely as the n^{th} power of the distance, and be attractive. Then at the distance r, the force $f = \frac{\mu}{r^n}$. Hence by equation (5), Sect. 1., we have

$$d.v^2 = 2fds = -\frac{2\mu dr}{r^n}.$$

Integrating this expression, we find

$$v^2 = \frac{2\mu}{n-1}\cdot\frac{1}{r^{n-1}} + C.$$

At the commencement of the motion, or when $v=0$, let $r=\Delta$; then

$$0 = \frac{2\mu}{n-1}\cdot\frac{1}{\Delta^{n-1}} + C.$$

Whence, subtracting, we obtain

$$v^2 = \frac{2\mu}{n-1}\cdot\left\{\frac{1}{r^{n-1}} - \frac{1}{\Delta^{n-1}}\right\} \qquad (1)$$

If we suppose the body to have fallen from infinity, $v^2 = \frac{2\mu}{n-1}\cdot\frac{1}{r^{n-1}}$. Hence the velocity from an infinite distance, Δ, to a finite distance, r, is finite, when the force varies inversely as any power of the distance higher than the first.

(21.) Let the force vary *directly* as the n^{th} power of the distance; and by proceeding in a similar manner, or by simply changing the sign of n in formula (1), we have

$$v^2 = \frac{2\mu}{n+1}\cdot\left\{\Delta^{n+1} - r^{n+1}\right\}. \qquad (2)$$

In this case, if the body fall from a finite distance, Δ, to the centre of force, we have a finite value, viz.

$$v^2 = \frac{2\mu}{n+1}\cdot\Delta^{n+1}.$$

In the former case, the velocity from a finite distance to the centre is *infinite*; and in the latter the velocity from infinity to a finite distance is *infinite*.

(22.) There is one case in which both the formulæ, (1) and (2), give indeterminate values. This is when the force is inversely as the distance. For this case we must return to the primitive differential equation, whence we have

$$d.v^2 = -2\mu.\frac{dr}{r}.$$

Hence

$$v^2 = -2\mu.\log.r + C.$$

Let $r=\Delta$ when $v=0$, and we have

$$v^2 = 2\mu.\log.\frac{\Delta}{r}.$$

This result proves that the velocity from an infinite to a finite distance is infinite, and also the velocity from a finite distance to the centre, when the force is inversely as the distance.

(23.) Our next object is to determine the time of the motion from any one distance to any other in the case of single centre, and a material point moving in a right line. This determination is not *generally* possible in finite terms, as shall be shewn further on. We shall proceed then to find the values of t in particular cases.

PROP. II.

(24.) *To determine the time of moving from any one distance to any other along a right line, the force being inversely as the square of the distance.*

Let $f=\frac{\mu}{r^2}$. By formula (1) in this case, $v=\frac{\sqrt{2\mu.(\Delta-r)}}{\sqrt{\Delta r}}$, but by equation (2), Sect. 1., $dt=\frac{ds}{v}$. Substituting for v, and using $(-dr)$ for ds, we obtain $dt=\sqrt{\frac{\Delta}{2\mu}}.\frac{-rdr}{\sqrt{\Delta r-r^2}}$.

To integrate this, assume $\sqrt{\Delta r-r^2}=\frac{1}{2}\Delta.\sin.\theta$, then

$$r=\tfrac{1}{2}\Delta.(1+\cos.\theta), \text{ and } dr=-\tfrac{1}{2}\Delta.\sin.\theta.d\theta..$$

Hence

$$dt=\sqrt{\frac{\Delta}{2\mu}}\cdot\left\{\tfrac{1}{2}\Delta.d\theta+\tfrac{1}{2}\Delta.\cos.\theta.d\theta\right\}\cdot$$

Integrating,

$$t=\sqrt{\frac{\Delta}{2\mu}}\cdot\left\{\tfrac{1}{2}\Delta.\theta+\tfrac{1}{2}\Delta.\sin.\theta\right\}+\mathrm{C}.$$

Let t be supposed to commence when $r=\Delta$, then when $t=0$, $\sin.\theta=0$, and $\cos.\theta=1$; and, therefore, $\theta=0$. Hence the constant $\mathrm{C}=0$. Therefore, the complete integral is

$$t=\frac{\Delta^{\frac{3}{2}}}{2\sqrt{2\mu}}\cdot\{\theta+\sin.\theta\} \qquad (3)$$

(25.) When $r=0$, $\sin.\theta=0$, and $\cos.\theta=-1$; hence $\theta=\pi$, and $t=\dfrac{\pi\Delta^{\frac{3}{2}}}{2\sqrt{2\mu}}\cdot$ The time from any distance, r', to any other, r'', may be found by taking the time through the spaces $(\Delta-r')$ and $(\Delta-r'')$, and taking their difference.

(26.) The reader may observe that in the preceding solution the velocity has been presumed to be less than that from infinity. If the velocity be that from infinity, we have

$$dt=\frac{-dr\sqrt{r}}{\sqrt{2\mu}},$$

and, therefore,

$$t=\frac{2}{3\sqrt{2\mu}}\cdot\left\{r'^{\frac{3}{2}}-r^{\frac{3}{2}}\right\}. \qquad (4)$$

This formula gives the time from any one distance, r', to any other, r.

(27.) Let the velocity be greater than that from infinity; and let then v^2 be written in the form $2\mu.\left(\dfrac{1}{r}+\dfrac{1}{a}\right)$. Hence

$$dt=\sqrt{\frac{a}{2\mu}}\cdot\frac{-r\,dr}{\sqrt{ar+r^2}}$$

Assume $\sqrt{ar+r^2}=\frac{1}{2}a.\tan.\theta$; then $r=\frac{1}{2}a.(\sec.\theta-1)$, and $dr=\frac{1}{2}a.d\theta.\tan.\theta.\sec.\theta$. Combining these values we have

$$dt=\frac{a^{\frac{3}{2}}}{2\sqrt{2\mu}}\cdot\{\sec.\theta d\theta-\sec.^2\theta d\theta\}.$$

Integrating,

$$t=\frac{a^{\frac{3}{2}}}{2\sqrt{2\mu}}\cdot\left\{\log.\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)-\tan.\theta\right\} \qquad (5)$$

This integral taken between any assigned limits will give the value of the time.

(28.) The integrals expressed by equations (3), (4), and (5), we shall exhibit by a construction as follows:

For equation (3) :—Let the line AB, (Fig. 25.), Part 1., be equal to Δ, and BC equal to r; then the time of falling through any space AC : time through AB :: sector ABD : semicircular area. For let BD$=z$, and the angle ABD$=\omega$; then we have $z^2=\Delta r$, and $z=\Delta\cos.\omega$; hence we can without difficulty obtain $\frac{z^2d\omega}{2}=-\frac{\Delta.r\,dr}{4\sqrt{\Delta r-r^2}}$; but $\frac{z^2d\omega}{2}$ is the increment of the sector ADB, which is therefore proportional to dt, and, therefore, since the sector and time commence together, they continue proportional.

For equation (4):—Computing similarly the increment of the parabolic segment BED, (Fig. 27.), Part 1., we shall find it proportional to $-dr.\sqrt{r}$, and, therefore, to dt.

For equation (5):—In (Fig. 26.), Part 1., denoting BC by r, and BD by z, we have, since the hyperbola is equilateral, $z^2=2ar+r^2$, and also we have $z.\cos.\omega=r$, from these equations, computing $\frac{z^2d\omega}{2}$, we find it proportional to $\frac{r\,dr}{\sqrt{ar+r^2}}$, and, therefore, to dt.

These constructions, the reader will observe, agree with the constructions of Newton, given in his seventh section, but obtained by a different method.

PROP. III.

(29.) *To determine the time when the force is inversely as the cube of the distance.*

Let $f=\frac{\mu}{r^3}$; then $dt=\frac{\Delta}{\sqrt{\mu}}\cdot\frac{-r\,dr}{\sqrt{\Delta^2-r^2}}\cdot$ Integrating this, we have

$$t=\frac{\Delta}{\sqrt{\mu}}\cdot\sqrt{\Delta^2-r^2}+C. \tag{6}$$

If $t=0$ when $r=\Delta$, then $C=0$.

For the whole time we have $t=\frac{\Delta^2}{\sqrt{\mu}}$.

PROP. IV.

(30.) *To determine the time when the force is directly as the distance.*

Let $f=\mu r$, then $v=\sqrt{\mu}.\sqrt{\Delta^2-r^2}$, whence

$$dt=\frac{-dr}{\sqrt{\mu}.\sqrt{\Delta^2-r^2}}.$$

Assume $\Delta.\sin.\theta=\sqrt{\Delta^2-r^2}$, then $\Delta.\cos.\theta=r$, and $\Delta.\sin.\theta.d\theta=-dr$. Combining these in the value of dt, and integrating, we have

$$t=\frac{1}{\sqrt{\mu}}\cdot\cos.^{-1}\frac{r}{\Delta}+C. \tag{7}$$

The constant C may be neglected if $t=0$ when $r=\Delta$.

Into the centre $t=\frac{\pi}{2\sqrt{\mu}}$.

PROP. V.

(31.) *To determine the time when the force is inversely as the square root of the distance.*

Let $f=\frac{\mu}{\sqrt{r}}$, then $v=2\sqrt{\mu}.\sqrt{\Delta^{\frac{1}{2}}-r^{\frac{1}{2}}}$: therefore, $dt=\frac{-dr}{2\sqrt{\mu}.\sqrt{\Delta^{\frac{1}{2}}-r^{\frac{1}{2}}}}$. Assume $\sqrt{\Delta}-\sqrt{r}=z^2$, then $\frac{-dr}{2\sqrt{r}}=2zdz$, and therefore, $-dr=4z.(\sqrt{\Delta}-z^2).dz$. Hence we have

$$dt=\frac{2}{\sqrt{\mu}}.(\sqrt{\Delta}-z^2).dz.$$

and

$$t=\frac{2}{\sqrt{\mu}}.\left(\sqrt{\Delta}.z-\frac{z^3}{3}\right).$$

or, replacing z by its value in terms of r,

$$t=\frac{2}{\sqrt{\mu}}.\sqrt{\Delta^{\frac{1}{2}}-r^{\frac{1}{2}}}.\left(\frac{2\Delta^{\frac{1}{2}}+r^{\frac{1}{2}}}{3}\right). \qquad (8)$$

Without multiplying instances any farther, we shall proceed to shew generally in what cases the expression for dt is an integrable function in finite terms.

PROP. VI.

(32.) *To investigate in what cases the general expression for dt is integrable in finite terms.*

The general value of v being substituted in the expression $dt=\frac{-dr}{v}$, we find

$$dt = -\sqrt{\frac{n-1}{2\mu} \cdot \Delta^{n-1}} \cdot \frac{r^{\frac{n-1}{2}} . dr}{\sqrt{\Delta^{n-1} - r^{n-1}}} .$$

This expression comes under the class of binomial differentials, and is, therefore, not integrable in finite terms, unless in particular cases.—*Vide* SIMPSON's *Fluxions*, *Art.* (221). LACROIX *Calcul. Diff. Art.* (193.), *Elementary Treatise.*

The general binomial, $x^{m-1}\,dx.\,(a+bx^n)^{\frac{p}{s}}$, is integrable (*vide* LACROIX *Elem. Treat. Art.* (192.)) when $\frac{m}{n}$ is an integer number. Hence the expression for dt is integrable when $\frac{n+1}{2} \cdot \frac{1}{n-1}$ is an integer. Let this integer be i, then $n = \frac{2i+1}{2i-1}$ in the integrable cases.

This comprises the two following series of values of n:—

for i negative, $\frac{1}{3}$, $\frac{3}{5}$, $\frac{5}{7}$, $\frac{7}{9}$, $\frac{9}{11}$, &c.

for i positive, -1, 3, $\frac{5}{3}$, $\frac{7}{5}$, $\frac{9}{7}$, &c.

The binomial $x^{m-1}dx.\,(a+bx^n)^{\frac{p}{s}}$ is also integrable when $\frac{m}{n}+\frac{p}{g}$ is an integer number; that is, if $\frac{n+1}{2(n-1)} - \frac{1}{s}$ be an integer, the value of dt is integrable; let this integer be i, then we have

$$n = \frac{1+i}{i} .$$

This comprises the following series of values of n:—

for i negative, 0, $\frac{1}{2}$, $\frac{2}{3}$, $\frac{3}{4}$, $\frac{4}{5}$, $\frac{5}{6}$, &c.

for i positive, ∞, 2, $\frac{3}{2}$, $\frac{4}{3}$, $\frac{5}{4}$, &c.

The only laws of force expressed by *integral* powers for which we can find the time, are

$$f=c,\quad f=\mu r,\quad f=\frac{\mu}{r^2},\ \text{and}\ f=\frac{\mu}{r^3}.$$

There is an infinite number of fractional powers.

PROP. VII.

(33.) *To find a series to express the time when the force varies inversely as the* n^{th} *power of the distance.*

The expression for dt may be written as follows:—

$$dt=-\sqrt{\frac{n-1}{2\mu}}\cdot r^{\frac{n-1}{2}}.dr.\left(1-\left(\frac{r}{\Delta}\right)^{n-1}\right)^{-\frac{1}{2}}$$

$$=-\sqrt{\frac{n-1}{2\mu}}\cdot r^{\frac{n-1}{2}}.dr.\left\{1+\tfrac{1}{2}\cdot\left(\frac{r}{\Delta}\right)^{n-1}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\left(\frac{r}{\Delta}\right)^{2n-2}+\ \&\text{c.}\right\}$$

Multiplying and integrating, we shall have

$$t=\sqrt{\frac{n-1}{2\mu}}\cdot\left.\begin{array}{l}\left\{C-\dfrac{2}{n+1}\cdot r^{\frac{n+1}{2}}-\tfrac{1}{2}\cdot\dfrac{2}{3n-1}\cdot\dfrac{r^{\frac{3n-1}{2}}}{\Delta^{n-1}}-\right.\\ \qquad\left.\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\dfrac{2}{5n-3}\cdot\dfrac{r^{\frac{5n-3}{2}}}{\Delta^{2n-2}}+\ \&\text{c.}\right\}\end{array}\right\}\quad(9)$$

If this be taken from $r=\Delta$ to $r=0$, it gives

$$t=\sqrt{\frac{n-1}{2\mu}}\cdot 2\,\Delta^{\frac{n+1}{2}}\left\{\frac{1}{n+1}+\tfrac{1}{2}\cdot\frac{1}{3n-1}+\tfrac{1}{2}\cdot\tfrac{3}{4}\cdot\frac{1}{5n-3}+\ \&\text{c.}\right\}\quad(10)$$

For the *same* law of force the whole time to the centre $\propto\Delta^{\frac{n+1}{2}}$.

There is one case for which series (10) will not answer, viz. when the force is inversely as the distance.

PROP. VIII.

(34.) *To determine the whole time of falling into the centre when the force is inversely as the distance.*

By Art. (22.) we have in this case $v = \sqrt{2\mu . \log . \frac{\Delta}{r}}$, therefore,

$$dt = -\frac{dr}{v} = -\frac{1}{\sqrt{2\mu}} . \frac{dr}{\sqrt{\log . \frac{\Delta}{r}}} .$$

Let $\frac{\Delta}{r} = e^{x^2}$, then $\log . \frac{\Delta}{r} = x^2$, and $dr = \frac{-2\Delta . dx}{e^{x^2}}$.

Hence

$$dt = \frac{2\Delta}{\sqrt{2\mu}} . \int e^{-x^2} dx.$$

In order to give the whole time, the value of t must be found from $r = \Delta$ to $r = 0$, that is, from $x = 0$ to $x = \infty$. The problem then belongs to the class of definite integrals, and it may be solved as follows. The quantity $\int \frac{z^{2n} dz}{\sqrt{1-z^2}}$ (*vide* LACR. *Calc. Diff. Elem. Art.* (197.) *Edit.* 3*d*) if taken between the limits $z = 0$ and $z = 1$, has for its value $\frac{1.3.5 \ldots\ldots (2n-1)}{2.4.6 \ldots\ldots 2n} . \frac{\pi}{2}$. And the quantity $\int \frac{z^{2n+1} dz}{\sqrt{1-z^2}}$ within the same limits is equal to $\frac{2.4.6 \ldots 2n}{3.5.7 \ldots 2n+1}$. Hence we have

$$\int \frac{z^{2n} . dz}{\sqrt{1-z^2}} . \int \frac{z^{2n+1} . dz}{\sqrt{1-z^2}} = \frac{1}{2n+1} . \frac{\pi}{2} .$$

Make $z = e^{-qx^2}$, and this expression may be written

$$4q^2.\int\frac{x.e^{-qx^2}.dx.e^{-2qnx^2}}{\sqrt{1-e^{-2qx^2}}}\times\int\frac{x.e^{-qx^2}.dx.e^{-q.(2n+1)x^2}}{\sqrt{1-e^{-2qx^2}}}=$$

$$\frac{1}{2n+1}.\frac{\pi}{2}.$$

For q in this writing $\frac{1}{2n+1}$ it may be reduced to the form

$$2.\int\frac{x.e^{-x^2}.dx}{\sqrt{\frac{1-e^{-2qx^2}}{2q}}}.\int\frac{x.e^{-x^2(1+q)}.dx}{\sqrt{\frac{1-e^{-2qx^2}}{2q}}}=\frac{\pi}{2}.$$

Now suppose $q=0$, and we obtain

$$2.\left\{\int e^{-x^2}dx\right\}^2=\frac{\pi}{2},\quad\text{or}\int e^{-x^2}dx=\pm\tfrac{1}{2}\sqrt{\pi}.$$

Hence we have

$$t=\frac{\Delta\sqrt{\pi}}{\sqrt{2\mu}}.$$

We shall next consider the case of a body moving in a right line passing through two centres of forces.

PROP. IX.

(35.) *To determine the rectilinear velocity of a material point attracted by two centres of force.*

Let the centres of force be A and B, and their distance l; let their respective intensities at the distances r and $l-r$ be $\frac{\mu}{r^n}$ and $\frac{\mu'}{(l-r)^n}$. Then

$$d.v^2=-2fds=2.\left\{\frac{\mu'dr}{(l-r)^n}-\frac{\mu dr}{r^n}\right\}.$$

Integrating this, and supposing that at the commencement of the motion $r=a$ and $v=$ A, we have

$$\left.\begin{array}{l} v^2 - A^2 = \frac{2\mu'}{n-1}\left\{\frac{1}{(l-r)^{n-1}} - \frac{1}{(l-a)^{n-1}}\right\} + \\ \qquad \frac{2\mu}{n-1}\cdot\left\{\frac{\mu}{r^{n-1}} - \frac{\mu}{a^{n-1}}\right\} \end{array}\right\} \quad (a)$$

This formula gives us the velocity at any point of the line when it is known for the point of departure.

(36.) At a certain point on the line joining the two centres their attractions are equal; to determine this point, let its distance from the centre A be r', then $\frac{\mu'}{(l-r'^n)} = \frac{\mu}{r'^n}$, whence

$$r' = l.\frac{\sqrt[n]{\mu}}{\sqrt[n]{\mu} + \sqrt[n]{\mu'}}.$$

As long as $r < r'$ the attraction of the centre A will exceed that of B, and when $r > r'$ that of B will exceed that of A. If $a < r'$, and the velocity k directed towards B, the velocity v may vanish before the body reaches the point of equal attraction C, in which case the body will fall back towards A. If the velocity vanish precisely at C, the body will remain in equilibrio at C. But if the velocity be sufficient to carry the body beyond C it will go on to B.

The value of k, just sufficient to move the body to C, may be found by supposing $v = 0$, and $r = r'$ in formula (a), and then solving for k.

SECTION III.

GENERAL THEOREMS DEDUCIBLE FROM THE DIFFERENTIAL EQUATIONS IN THE CASE OF CURVILINEAR MOTION.—HYPOTHESIS OF A SINGLE CENTRE.—DEDUCTION OF FORMULÆ FOR THE DIFFERENT AFFECTIONS OF A MATERIAL POINT SUBJECTED TO THE ACTION OF SUCH A CENTRE.

(37.) The three following differential equations of curvilinear motion have been proved in Art. (16.), Sect. 1.:—

$$\frac{d^2x}{dt^2}=\mathrm{X},\quad \frac{d^2y}{dt^2}=\mathrm{Y},\quad \frac{d^2z}{dt^2}=\mathrm{Z}. \qquad (1)$$

These equations require for their complete integration the determination of six arbitrary constants for any particular values of X, Y, Z. These constants are to be determined by the initial conditions of the motion; such as, for instance, the velocity, the direction of projection, and the position of the moving point at the commencement of the motion. The question would become simplified were we to suppose the moving point restricted to a given plane, as we might then omit the consideration of one of the axes. Even with this simplification their integration in most cases presents insurmountable difficulties. We shall evade this question for the present, and proceed to determine some general results which hold true exclusive of any particular law of force.

PROP. X.

(38.) *To determine the velocity acquired by a material point in moving from one point of space to another.*

Multiplying the first of equations (1) by $2dx$, the second by $2dy$, and the third by $2dz$, and adding all three together we obtain

$$\frac{2dx\,d^2x+2dy.\,d^2y+2dz\,d^2z}{dt^2}=2.\,(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz).$$

From this, by integrating, we obtain

$$\frac{dx^2+dy^2+dz^2}{dt^2}=\beta+2.\int(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz).$$

The left hand member of this equation expresses the square of the velocity, since $dx^2+dy^2+dz^2=ds^2$. The velocity can therefore be found in any state of the co-ordinates, when the right hand member can be integrated. This integration cannot be effected unless in three cases, viz. when X, Y, Z, vanish, are constant, or are functions of x, y, z; for should they involve any other variables, the function $(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$ would involve more variables than differentials, and would, therefore, be essentially unintegrable.

In the integrable cases we have

$$\left.\begin{aligned} v^2&=\beta\\ v^2&=\beta+2.(\mathrm{X}x+\mathrm{Y}y+\mathrm{Z}z)\\ v^2&=\beta+2f.(x,y,z)\end{aligned}\right\}\quad(2)$$

In this last equation $f.(x, y, z)$ expresses the found integral.

In order to find β, let A^2 be the square of the velocity when the co-ordinates have the values a, b, c, we have then

$$\mathrm{A}^2=\beta+2.f.(a,b,c).$$

Hence eliminating β, we have

$$v^2-\mathrm{A}^2=2.f.(x,y,z)-2f.(a,b,c).\qquad(3)$$

In curves that return into themselves, the velocity is always the same at the same point; for x, y, z become of their former value, whence $v^2-A^2=0$.

(39.) In the case of a single centre, let F be the force directed thereto. Then making this centre the origin of co-ordinates, we have $X=F.\frac{x}{r}$, $Y=F.\frac{y}{r}$, $Z=F.\frac{z}{r}$, where $r=\sqrt{x^2+y^2+z^2}$. By differentiating, we find $dr=\frac{x}{r}\cdot dx+\frac{y}{r}\cdot dy+\frac{z}{r}\cdot dz$; therefore,

$$Fdr=F.\frac{x}{r}\cdot dx+F.\frac{y}{r}\cdot dy+F.\frac{z}{r}\cdot dz=Xdx+Ydy+Zdz.$$

Hence we have

$$v^2=\beta+2\int Fdr.$$

Now let F be a function of r, then $\int Fdr$ can be found: let its value be $f(r)$, and we find

$$v^2-A^2=2f(r)-2f(r').$$

Hence, if bodies be revolving in different orbits about the same centre, and have the same velocities at any instance of equal altitudes, they shall have the same velocities at all instances of equal altitudes.—*Newt. Prin. Pr.* 40. *sect.* 8.

(40.) If $F=\mu.r^n$, then $v^2-A^2=\frac{2\mu}{n+1}\cdot\left(r^{n+1}-r'^{n+1}\right)$, and therefore $d.v^2 \propto r^n dr$. Hence if a curve be described whose abscissæ are the values of r, and the ordinates of which $\propto r^n$, the areas of this curve will represent the squares of the velocities. The representative curve is of the hyperbolic species if n be negative, and of the parabolic if n be positive.

PROP. XI.

(41.) *The expression* $Xdx+Ydy+Zdz$ *is always integrable when the moving point is acted on by forces directed to any number of fixed centres.*

Let the co-ordinates of one of these centres be a, b, c; and r the distance from it to the moving point. Let F be the force directed towards this centre, then the components of this force parallel to the axes of x, y, z respectively, are $F.\frac{x-a}{r}$, $F.\frac{y-a}{r}$, $F.\frac{z-a}{r}\cdot$ Since $r^2=(x-a)^2+(y-b)^2+(z-c)^2$, by differentiating we find

$$dr=\frac{x-a}{a}\cdot dx+\frac{y-b}{r}\cdot dy+\frac{z-c}{r}\cdot dz;$$

and multiplying by F,

$$F.dr=F.\frac{x-a}{r}\cdot dx+F.\frac{y-b}{r}\cdot dy+F.\frac{z-c}{r}\cdot dz.$$

Similarly for another centre we have

$$F'dr'=F'.\frac{x-a'}{r'}\cdot dx+F'.\frac{y-b'}{r'}\cdot dy+F'.\frac{z-c'}{r'}\cdot dz.$$

Hence we have in general

$$Xdx+Ydy+Zdz=Fdr+F'dr'+\&c.$$

(42.) If the centres be in motion this is not the case, for then a, b, c are variables.

(43.) If the whole system be in motion with a common velocity, the function continues integrable; for a, b, c are constant relative to the moving origin.

(44.) The most remarkable cases in which the function cannot be integrated, are when the acting forces are directed towards centres in motion, as has been already noticed; or when there are amongst the forces any arising from friction, or the resistance of an inert fluid.

PROP. XII.

(45.) *If a material point be acted on by a force directed to a single centre, the projections on each of three rectangular planes, of the areas it describes, are proportional to the times.*

If the centre of force be made the origin of co-ordinates, and F be the whole force directed thereto, it is obvious that the components of F along the three axes are to each other respectively as x, y, z. We have then $X : Y : Z :: x : y : z$.

Multiply the first of equations (1) by y, and the second by x, and subtract them, there results

$$\frac{yd^2x-xd^2y}{dt}=(Xy-Yx).\,dt=0.$$

Integrating this expression we obtain

$$ydx-xdy=hdt. \qquad (4)$$

Similarly,

$$xdz-zdz=h'dt$$

$$zdy-ydz=h''dt.$$

Now let st (Fig. 44.) be the orbit in space, which the moving point passes over; let $s't'$ be the same orthographically projected on the plane of xy; let $ot'=\rho$, and the angle $t'ox=\omega$; then $x=\rho.\cos.\omega$, $y=\rho.\sin.\omega$. Hence differentiating we obtain the values of dx and dy, and thence find $ydx-xdy=\rho^2 d\omega$, but $\rho^2 d\omega$ is double the sector $t'os'$ traced on the plane of xy. Hence the equations (4) express the proportionality of these areas to the times.

(46.) Independently of the particular case just mentioned, there are two general cases in which the equations (4) hold true:—1st, when the forces X, Y, Z vanish, and that the body moves in consequence of a primitive impulse;—2d, When the system consists of any number of bodies in free space subjected to their mutual attraction, the sum of the products which arise by multiplying the mass of each body into the area traced out by it, is proportional to the time. This shall be proved further on.

(47.) The peculiarity in these two latter cases is, that proportionality of the areas to the times holds true for any point in space; whereas in the case mentioned in Art. (45.) it only holds for the centre of force.

PROP. XIII.

(48.) *When the system consists of a single body revolving about a fixed centre of force, the curve described lies altogether in a plane passing through this fixed centre.*

Multiply the first of equations (4) by z, the second by y, and the third by x, and add them. The result is

$$hz+h'y+h''x=0. \qquad (5)$$

which is the equation of a plane through the origin of co-ordinates.

PROP. XIV.

(49.) *To transform the three differential equations of motion to three others in which the forces shall be; a force in the plane of x, y directed to the origin of co-ordinates; a force in the same plane perpendicular thereto; and a force perpendicular to the plane of x, y.*

Let ρ be the *radius vector* in the plane of x, y, or the distance from the origin to the point of projection; and and let ω be the angle which ρ makes with the axis of x. Let F be the force along ρ, and T the force perpendicular to it in the plane of xy. By decomposition of forces we have

$$\mathrm{X} \cos. \omega + \mathrm{Y} \sin. \omega = \mathrm{F}$$

$$\mathrm{Y} \cos. \omega - \mathrm{X} \sin. \omega = \pm \mathrm{T}.$$

Since $x = \rho. \cos. \omega$ and $y = \rho. \sin. \omega$, we have, by differentiating,

$$d^2x = d^2\rho. \cos. \omega - 2\, d\omega. d\rho. \sin. \omega - \rho. \cos. \omega. d\omega^2 - \rho. d^2\omega. \sin. \omega$$

$$d^2y = d^2\rho. \sin. \omega + 2\, d\omega. d\rho. \cos. \omega - \rho. \sin. \omega\, . d\omega^2 + \rho d^2\omega. \cos. \omega.$$

These equations give us

$$d^2x. \cos. \omega + d^2y. \sin. \omega = d^2\rho - \rho\, d\omega^2$$

$$d^2y. \cos. \omega - d^2x. \sin. \omega = 2\, d\omega. d\rho + \rho\, d^2\omega.$$

From these by equations (1) we obtain

$$\left.\begin{array}{l} d^2\rho - \rho\, d\omega^2 = \mathrm{F}. dt^2 \\ 2\, d\omega. d\rho + \rho. d^2\omega = \pm \mathrm{T}. dt^2. \end{array}\right\} (6)$$

If s denote the tangent of the angle contained by the line r to the moving point and its projection ρ, the third differential equation may be written

$$d^2(\rho s) = \mathrm{Z}. dt^2.$$

(50.) If the moving body were a planet, and the plane of x, y the plane of the ecliptic; the line ρ would be called the curtate distance, and s the tangent of the latitude. The equations (6), though inferior in compactness and apparent simplicity to equations (1), are still decidedly superior in this, that when once solved they directly exhibit the values of ρ, ω, and s, quantities of immediate use in Astronomical enquiries.

(51.) In the case of a force directed to a single centre, from the second of the above equations making $T=0$, we obtain by integration $\rho^2 d\omega = h dt$, which proves the equable description of areas.—*See Art.* (45).

PROP. XIV.

(52.) *In the case of an attractive or repulsive force directed to a single centre to investigate a general value for the force.*

As the motion of the body is by Art. (48.) confined to a plane, we may take this plane for the plane of x, y, and omit the consideration of a third axis. Let F be the force directed to the centre selected as origin of co-ordinates: then the components of F parallel to the axes of x and y respectively, are $F.\frac{x}{r}$, $F.\frac{y}{r}$. Hence by equations (1) we have

$$\frac{d^2x}{dt^2} = \mp F.\frac{x}{r}. \qquad \frac{d^2y}{dt^2} = \mp F.\frac{y}{r}.$$

The sign is (—) or (+), according as F is attractive or repulsive, for in the former case the force tends to diminish the co-ordinates x, y, and in the latter to increase them. Multiply these equations, the first by $2dx$, and the second by $2dy$, and add them. The result obtained being integrated gives us

$$\frac{dx^2+dy^2}{dt^2}=\beta\mp 2\int \mathrm{F}dr.$$

Substituting for dx^2+dy^2 its value $dr^2+r^2d\omega^2$, and eliminating dt by the equation $r^2d\omega=hdt$, we obtain

$$\frac{h^2(dr^2+r^2d\omega^2)}{r^4d\omega^2}=\beta\mp 2\int \mathrm{F}dr.$$

Differentiating this we find

$$\mathrm{F}dr=\mp\frac{h^2}{2}\cdot d.\frac{dr^2+r^2d\omega^2}{r^4d\omega^2}. \qquad (7)$$

This very general formula comprises the several expressions for F that have been given by different writers. This shall appear by what follows.

PROP. XV.

(53.) *To investigate an expression for the force in terms of the perpendicular from the centre of force on the tangent, and the radius vector.*

Denoting by θ the angle contained by the tangent and radius vector, we have obviously $\sin.^2\theta=\frac{r^2d\omega^2}{dr^2+r^2d\omega^2}$, but $\sin.^2\theta=\frac{p^2}{r^2}$, therefore, $\frac{1}{p^2}=\frac{dr^2+r^2d\omega^2}{r^4d\omega^2}$. Hence by equation (7) we have

$$\mathrm{F}=\pm\frac{h^2dp}{p^3dr}. \qquad (8)$$

This formula has been very generally used by English writers. *Vide* SIMPSON'S *Fluxions, Art.* (226.); DEALTRY'S *Fluxions, Art.* (152.), &c.

PROP. XVI.

(54.) *To investigate an expression for the force in terms of the velocity and chord of curvature.*

Let v denote the velocity of the moving point along the trajectory, then the elementary area $=\frac{1}{2}v.p$, hence $h=v.p$; substituting this value in the preceding formula, we obtain

$$F=\pm\frac{2v^2dp}{2pdr},$$

but the chord of curvature (which we shall denote by C) is equal to $\frac{2pdr}{dp}$. Hence we have

$$F=\pm\frac{2v^2}{C}. \qquad (9)$$

PROP. XVII.

(55.) *To determine an expression for the force in terms of the sagitta and time.*

The velocity v being always equal to the increment of the space divided by the increment of the time, we have

$$v=\frac{(d.\,\text{arc})}{dt},$$

and therefore

$$F=\pm\frac{2.\,(d.\,\text{arc})^2}{C.\,dt^2},$$

but $\frac{(d.\,\text{arc})^2}{C}$ is the *sagitta* or deflection from the tangent. We have therefore

$$F=\pm\frac{2.\,(\text{sagitta})}{dt^2}. \qquad (10)$$

This is the expression with which Newton commences.—*Prin. Sect. 2. Pr. 6.*

(56.) The preceding expressions for F may be still further modified; for instance, since $C=\frac{2p\,dr}{dp}$, by formula (8),

$$F=\pm\frac{2h^2}{p^2C}\cdot \qquad (11)$$

Or writing $r^2\sin.^2\theta$ for p^2, we have

$$F=\pm\frac{2h^2}{r^2\sin.^2\theta.\,C}\cdot \qquad (12)$$

Let p' be the length of a perpendicular from the adjoining point of the curve on r, then $p'r=h\,dt$. Hence formula (10) may be written

$$F=\pm\frac{2h^2s}{r^2p'^2}\cdot \qquad (13)$$

Having thus accounted for the variety of formulæ that have been at different times given for the law of force, we shall next proceed to deduce the value of F which directly and naturally follows from formula (7).

PROP. XVIII.

(57.) *To investigate an expression for the force in terms of the radius vector r and the angle ω made with a given axis.*

By formula (7) we have $F\,dr=\mp\frac{h^2}{2}\cdot d.\left\{\frac{dr^2}{r^4d\omega^2}+\frac{1}{r^2}\right\}$, from which we can immediately deduce

$$F\,dr=\pm\frac{h^2}{2}\left\{\frac{dr}{r^3}-d.\frac{dr^2}{2\,r^4d\omega^2}\right\}\cdot \qquad (14)$$

(58.) The first part of this value of F expresses the force that a body should be acted on by to describe a circle at the same distance, and with the same elementary area, or angular velocity. The force in a central circle having been designated since the time of Huyghens by the name of centrifugal force, this name has in consequence been extended to that part of every force, sufficient to cause the body to revolve in a circle.

(59.) Let ϕ denote the centrifugal force; then $\phi=\frac{h^2}{r^3}$. Combining this with formula (12), we have

$$\frac{F}{\phi}=\frac{2r}{C.\sin.^2\theta}. \qquad (15)$$

(60.) The other part of the force is employed in accelerating or retarding the motion of the body in the direction of the *radius vector*. The whole velocity in this direction is called the *paracentric velocity*.

PROP. XIX.

(61.) *To investigate an expression for the paracentric velocity.*

Let F′ be the force producing it, then $d.\frac{dr^2}{2r^4d\omega^2}=F'dr$, hence

$$\frac{h^2dr^2}{r^4d\omega^2}=\beta'+2\int F'dr.$$

But by Art. (39.) $\beta'+2\int F'dr$ expresses the square of the velocity produced by the action of the force F′. Hence we have, (denoting the paracentric velocity by the initials P. v),

$$(P.v)^2=\frac{h^2dr^2}{r^4d\omega^2}. \qquad (16)$$

Or thus:—$(P.v)^2 = v^2.\cos.^2\theta$, but $v^2 = \frac{h^2}{r^2.\sin.^2\theta}$, hence

$$(P.v)^2 = \frac{h^2}{r^2}.\frac{1}{\tan.^2\theta};$$

but $\tan.^2\theta = \frac{r^2 d\omega^2}{dr^2}$, and therefore,

$$(P.v)^2 = \frac{h^2 dr^2}{r^4 d\omega^2}:$$

(62.) At the point where the paracentric velocity is greatest or least, the second term of formula (14) vanishes; therefore, at this point the centripetal and centrifugal forces are equal. The converse is not necessarily true.

PROP. XX.

(63.) *To investigate a formula for the velocity in the orbit, in terms of the force.*

By formula (9) we have $F = \pm\frac{2v^2}{C}$, hence

$$v = \sqrt{\pm\tfrac{1}{2}F.C.} \qquad (17)$$

When F is repulsive, C must be negative, or the curve convex to the centre of force.

(64.) With the force F acting uniformly, the velocity acquired in moving through the space Σ is by Art. (11.) equal to $\sqrt{2F\Sigma}$. This would be equal to the velocity in the curve if $\Sigma = \frac{1}{4}C$. *Hence the velocity at any point of a curve is that which would be acquired in moving through the fourth part of the chord of curvature, with the force at that point continued uniform.*

(65.) Let v' be the velocity in a circle whose centre is the centre of force, and radius the distance of any point of the orbit; if this circle were described by the action of the same centre, we would have

$$v' = \sqrt{\tfrac{1}{2}\mathrm{F}.2r}.$$

Hence

$$\frac{v}{v'} = \sqrt{\pm\frac{\mathrm{C}}{2r}} = \sqrt{\pm\frac{p\,dr}{r\,dp}}. \tag{18}$$

PROP. XXI.

(66.) *To investigate a formula for the angular velocity.*

The angular velocity means the ratio of the increment of the angle described about the centre of force to the increment of the time. Denoting it by a, we have

$$a^2 = \frac{d\omega^2}{dt^2} = \frac{h^2}{r^4} = \frac{v^2p^2}{r^4}. \tag{19}$$

(67.) Let a' denote the angular velocity in the equidistant circle described by the action of the same centre. We have then

$$\frac{a^2}{a'^2} = \frac{v^2}{v'^2}\cdot\frac{p^2}{r^2} = \frac{\mathrm{C}.\sin.^2\theta}{2r}. \tag{20}$$

PROP. XXII.

(68.) *To investigate an expression whence the time of describing any portion of the orbit may be found.*

By the equation $r^2d\omega = h\,dt$, we have

$$t = \int\frac{r^2d\omega}{h}. \tag{21}$$

The whole periodic time might be found if we knew the area of the orbit, by dividing this area by h. This supposes, however, that we know h. The value of h is found by knowing the absolute force or force at the unit of distance. Let us denote this absolute force by μ, then the force F at any distance is equal to $\mu . \phi (r)$; but by formula (14) $F = A . h^2 \phi (r)$, therefore h can be found.

PROP. XXIII.

(69.) *To determine from what point a body must commence a rectilinear motion towards the orbit, so that arriving at it the body may have the velocity in that point of the orbit.*

Let the required distance be Δ, and let r be the distance of the point of the orbit, each from the centre of force; let $F = \pm \frac{\mu}{r^n}$: then by formula (17) the square of the velocity in the orbit is equal to $\pm \frac{\mu C}{2 r^n}$. But the square of the velocity in moving from Δ to r, by formula (1), Chap. II., is $\frac{2\mu}{n-1} . \left\{ \frac{1}{r^{n-1}} - \frac{1}{\Delta^{n-1}} \right\}$. Equating these values we find

$$\frac{2}{\Delta^{n-1}} = \frac{2}{r^{n-1}} \mp \frac{(n-1) . C}{2 r^n} = \frac{2}{r^{n-1}} \mp \frac{n-1}{r^n} . \frac{p dr}{dp} . \qquad (22)$$

From this formula we can determine Δ for any assigned law of force; or as it is termed, we can find how far a body must fall externally to acquire the velocity in a curve.

PROP. XXIV.

(70.) *To determine how far a body must fall from the curve with a rectilinear motion to acquire the velocity at the point of the curve whence it departed.*

Let Δ' be the distance required, then we have

$$\pm\frac{\mu C}{2r^n}=\frac{2\mu}{n-1}\cdot\left\{\frac{1}{\Delta'^{n-1}}-\frac{1}{r^{n-1}}\right\}.$$

Hence

$$\frac{2}{\Delta'^{n-1}}=\frac{2}{r^{n-1}}\pm\frac{(n-1).C}{2r^n}=\frac{2}{r^{n-1}}\pm\frac{n-1}{r^n}\cdot\frac{pdr}{dp}. \qquad (23)$$

These formulæ can be adapted to a direct law of force by changing the sign of n.

(71.) By adding together the equations (22) and (23), it appears that the $(n-1)^{\text{th}}$ powers of Δ, r, and Δ' are in harmonic progression. And the three quantities Δ, r, and Δ' are themselves in harmonic progression when $n=2$.

For a direct law of force, or an inverse one less than the first, we find by adding the same equations that the $(n+1)^{\text{th}}$ powers of Δ, r, and Δ' are in Arithmetic progression. Unless $n=0$ the quantities Δ, r, and Δ' are not in Arithmetic progression.

(72.) The formulæ (22) and (23) will not answer when $n=1$. For this case we have $\frac{\mu C}{2r}=\mu.\log.\frac{\Delta}{r}$; whence for the space fallen through *externally*, we find

$$\Delta=r.\,\varepsilon^{\frac{pdr}{rdp}};$$

ε being the base of Napier's system of logarithms.

For the space fallen through *internally*, we have

$$\frac{\mu C}{2r}=\mu.\log.\frac{r}{\Delta'};$$

whence we have

$$\Delta'=r.\,\varepsilon^{\frac{-pdr}{rdp}}.$$

In this case the three quantities Δ, r, and Δ' are in geometric progression.

PROP. XXV.

(73.) *To investigate a formula for the element of the velocity along the curve.*

The accelerating force along the curve is equal to the central force multiplied by the cosine of the angle of projection. By formula (8) $F=\frac{h^2dp}{p^3dr}$; and $\cos.\theta=\sqrt{1-\frac{p^2}{r^2}}$. Hence the increment of the velocity, being proportional to the force

$$d.v=\frac{h^2dp}{p^3dr}\cdot\sqrt{1-\frac{p^2}{r^2}}\cdot dt \qquad (24)$$

PROP. XXVI.

(74.) *To investigate a formula for the increment of the angular velocity.*

The increment of the angular velocity is equal to the accelerating force perpendicular to r, multiplied by dt, and divided by r. The force along the curve is to the force perpendicular to r, as r to p: hence the force perpendicular to r is equal to

$$\frac{h^2dp}{p^2rdr}\cdot\sqrt{1-\frac{p^2}{r^2}}.$$

Hence

$$d.a=\frac{h^2dp}{p^2r^2dr}\cdot\sqrt{1-\frac{p^2}{r^2}}\cdot dt. \qquad (25)$$

PROP. XXVII.

(75.) *To find the polar equation to the curve described, in terms of the reciprocal of the radius vector, and the angle it makes with a fixed axis.*

By formula (7), Art. (52.), we have

$$\mathrm{F}dr = \pm\frac{h^2}{2}\cdot d.\frac{dr^2+r^2d\omega^2}{r^4d\omega^2}.$$

Make $u=\frac{1}{r}$, then $dr=-\frac{du}{u^2}$, substituting then for r and dr in terms of u, we have

$$\frac{\mathrm{F}du}{u^2} = \pm\frac{h^2}{2}\cdot d.\left\{\frac{du^2}{d\omega^2}+u^2\right\}.$$

Performing the required differentiation, supposing $d\omega$ constant, and arranging, the result is

$$\frac{d^2u}{d\omega^2}+u\pm\frac{\mathrm{F}}{h^2u^2}=0. \qquad (26)$$

This formula we shall employ for the determination of the orbit, when given the law of force.

PROP. XXVIII.

(76.) *To investigate a formula for comparing the angular velocity in the curve with that of the perpendicular on the tangent.*

The angular velocity of the perpendicular on the tangent is equal to that of the radius of curvature about the centre of curvature. But the angular velocity in the curve is to that about the centre of curvature directly as the subtending arcs and inversely as the distances. Let a denote the angular velocity of the perpendicular on the tangent, or of the radius of curvature: then $\frac{a}{a}=\frac{\text{A}}{\text{A}'}\cdot\frac{r}{\gamma}$. But $\frac{\text{A}}{\text{A}'}=\frac{2\gamma}{\text{C}}$, therefore, $\frac{a}{a}=\frac{2r}{\text{C}}$. Hence

$$a=\frac{h}{r^2}\cdot\frac{2r}{\text{C}}=\frac{2h}{r\text{C}}.$$

We shall next proceed to apply the formulæ hitherto established to the discovery of the laws of force requisite, that different curves may be described about fixed centres of force, and also to the discovery of the curves themselves, the laws of force and conditions of the motion being assigned.

SECTION IV.

ON THE INVESTIGATION OF LAWS OF FORCE, VELOCITIES, &C. IN SEVERAL DIFFERENT CURVES, BOTH ALGEBRAICAL AND TRANSCENDENTAL.

(77.) In the application of the general principles laid down in the preceding section to the theory of curvilinear motion, there are two classes of problems which present themselves to our notice. The first is, when given the curve and the position of the centre of force in its plane, it is required to investigate the law of force, velocity, &c. which a material point must have in order to move therein. The second, when given the law of force, velocity, &c. it is required to investigate the *nature* and *form* of the orbit described by a material point under the influence of such. The former is called the *direct*, and the latter the *inverse* methods of central forces.

These two methods of central forces comprise each two species of problems which we shall treat in their order.

I. When the centre of force, though in the plane of the orbit, is supposed to be infinitely remote, so that the force acts along a system of parallel lines.

II. When the centre of force is at a finite distance, and consequently the lines drawn thereto not parallel.

§ I.

PROP. XXIX.

(78.) *To investigate the law of force, velocity, &c. when the orbit described is any conic section, and the force directed to an infinitely distant centre.*

Equation (1) of the preceding section will readily solve this and all such problems. Let R denote the force, and let it act along the axis of y, then $\frac{d^2y}{dt^2} = \mp \text{R}$, and $\frac{d^2x}{dt^2} = 0$. This latter gives us

$$\frac{dx}{dt} = c,$$

which shews that the velocity along the axis of x is uniform: and the former, by eliminating dt, becomes

$$\frac{c^2 . d^2y}{dx^2} = \mp \text{R}.$$

The general equation of the conic sections gives us

$$y = \frac{-(\text{B}x + \text{D}) \pm \sqrt{(\text{B}^2 - 4\,\text{AC})\,x^2 + 2\,(\text{BD} - 2\,\text{AE})\,x + \text{D}^2 - 4\,\text{AF}}}{2\,\text{A}}.$$

Hence

$$2\,\mathrm{A}.\frac{dy}{dx}=-\mathrm{B}\pm\frac{(\mathrm{B}^2-4\,\mathrm{AC}).\,x+\mathrm{BD}-2\,\mathrm{AE}}{\surd^{\overline{\quad}}}.$$

Differentiating a second time, we obtain

$$2\,\mathrm{A}.\frac{d^2y}{dx^2}=\pm\frac{(\mathrm{B}^2-4\mathrm{AC}).(\mathrm{D}^2-4\mathrm{AF})-(\mathrm{BD}-2\mathrm{EA})^2}{(\surd^{\overline{\quad}})^3}.$$

but $\surd^{\overline{\quad}}=2\mathrm{A}y+\mathrm{B}x+\mathrm{D}$, hence

$$\frac{d^2y}{dx^2}=\frac{(\mathrm{B}^2-4\mathrm{AC})\,(\mathrm{D}^2-4\mathrm{AF})-(\mathrm{BD}-2\mathrm{AE})^2}{2\mathrm{A}.(2\mathrm{A}y+\mathrm{B}x+\mathrm{D})^3}.$$

The force along the ordinates of any conic section is therefore inversely as the cube of the quantity $(2\mathrm{A}y+\mathrm{B}x+\mathrm{D})$.

(79.) If the axis of x be a diameter, and the axis of y parallel to the ordinates, B and D both vanish, and the force is therefore inversely as the cube of the ordinate. Its actual value is $\mathrm{R}=-\frac{c^2p^2}{y^3}$, where p denotes the parameter to the diameter coinciding with the axis of x.

(80.) The velocity along the axis of x being uniform, the velocity along the orbit may be found by dividing by the sine of the angle under the tangent and ordinate, and multiplying by the sine of the angle under the axes of co-ordinates.

The time is found for any value of x by dividing this value of x by c.

(81.) If the axes of x and y be asymptotes to an hyperbola, A and D both vanish, and the force varies inversely as x^3, or directly as y^3. Its value is $\mathrm{R}=\frac{2\,y^3}{m^4}$, where $m^2=xy$.

(82.) If the axis of y be a diameter of a parabola A and B, both vanish, and the force is therefore constant.

PROP. XXX.

(83.) *If the force act along the ordinates to the base of a cycloid to determine its law.*

The origin being taken at the middle of the base, the differential equation of the cycloid is

$$\frac{dy}{dx}=-\sqrt{\frac{a-y}{y}}.$$

Differentiating this a second time, we obtain

$$\frac{c^2d^2y}{dx^2}=-\frac{2\,ac^2}{y^2}=-\frac{\mu}{y^2}.$$

Hence the force varies inversely as the square of the ordinate.

(84.) Since the velocity is equal to the constant velocity *c*, divided by the sine of the angle under the ordinate and tangent, we have

$$v=\frac{c.\sqrt{a}}{\sqrt{y}}=\sqrt{\frac{\mu}{2y}}.$$

(85.) The time is equal to $\frac{\pi a}{c}=\frac{a^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}$.

§ II.

PROP. XXXI.

(86.) *To investigate the law of force, velocity, &c. in the conic sections, the focus being the centre of force.*

The polar equation of the ellipse relative to the focus is

$$r=\frac{a.(1-e^2)}{1+e\cos.\omega},\qquad \text{or,}\ \frac{1}{r}=\frac{1}{a(1-e^2)}+\frac{e.\cos.\omega}{a(1-e^2)}.$$

Hence

$$\frac{dr^2}{r^4d\omega^2}=\frac{e^2.\sin.^2\omega}{a^2(1-e^2)^2}.$$

Differentiating, we have

$$d.\frac{dr^2}{2r^4d\omega^2}=\frac{e^2.\sin.\omega.\cos.\omega.d\omega}{a^2.(1-e^2)^2}=\frac{dr}{r^2}.\left\{\frac{1}{r}-\frac{1}{a(1-e^2)}\right\}.$$

Substituting this value in formula (14), Art. (57.), we have

$$F=\frac{h^2}{a.(1-e^2)}.\frac{1}{r^2}.$$

If the orbit be an hyperbola, $F=\pm\frac{h^2}{a(e^2-1)}.\frac{1}{r^2}$.

In the parabola, let 4 D denote the principal parameter, and we have

$$F=\frac{h^2}{2D}.\frac{1}{r^2}.$$

Hence, in general, in the focal conic sections the force varies inversely as the square of the distance from the focus.

(87.) Let μ denote the absolute force or force at the unit of distance, and the value of F in general is $F=\pm\frac{\mu}{r^2}$.

Since $\mu=\frac{h^2}{\frac{1}{2}L}$, we have in general $h=\sqrt{\frac{1}{2}L.\mu}$, L denoting the principal parameter.

(88.) *To compare the centripetal and centrifugal forces.*—In the ellipse and hyperbola the chord of curvature through the focus is $\frac{2b'^2}{a}$, and $\sin.^2\theta=\frac{b^2}{b'^2}$, therefore, from formula (15), Art. (58.),

$$\frac{F}{\phi}=\frac{r}{a.(1-e^2)}.$$

In the parabola $C=4D'$, and $\sin.^2\theta=\frac{D}{D'}$, therefore,

$$\frac{F}{\phi}=\frac{r}{2D}.$$

F and ϕ are equal in all the conic sections at the extremity of the focal ordinate. In the ellipse $F > \phi$ through the remote apsis, and $< \phi$ through the near one.

(89.) *To determine the paracentric velocity.*—From formula (16), Art. (61.),

In the ellipse, $(P.v.)^2=\frac{h^2e^2.\sin.^2\omega}{2a^2(1-e^2)^2}=\frac{\mu e^2.\sin.^2\omega}{2a.(1-e^2)}$.

In the hyperbola, $(P.v.)^2=\frac{h^2e^2.\sin.^2\omega}{2a^2(e^2-1)^2}=\frac{\mu e^2.\sin.^2\omega}{2a.(e^2-1)}$.

In the parabola, $(P.v.)^2=\frac{h^2.\sin.^2\omega}{8D^2}=\frac{\mu.\sin.^2\omega}{4D}$.

Hence all the variations of the paracentric velocity are obvious: *e.g.* it vanishes at ends of the axis major, or as they are called the *apsides*, and it becomes a *maximum* at the extremities of the focal ordinate, as also appears from Arts. (62.) and (57.).

(90.) *To determine the velocity.*—By Art. (63), putting for F and C their values, we have

In the ellipse, $v^2=\mu.\frac{2a-r}{ar}$,

In the hyperbola, $v^2=\mu.\frac{2a+r}{ar}$,

In the parabola, $v^2=\frac{2\mu}{r}$.

Hence in the ellipse the velocity at the distance a is a geometric mean between the greatest and least.

(91.) *To compare the velocity with that in the equidistant circle.*—We have by Art. (65.),

In the ellipse, $\frac{v^2}{v'^2}=\frac{2a-r}{a}$,

In the hyperbola, $\frac{v^2}{v'^2}=\frac{2a+r}{a}$,

In the parabola, $\frac{v^2}{v'^2}=2$.

Hence in the ellipse at the distant apse the velocity is less than in the equidistant circle, and therefore the revolving body descends towards the centre of force; and at the nearer apse it ascends, the velocity being greater than in the equidistant circle. At mean distance the velocity is equal to that in the equidistant circle, but the direction of the motion not being perpendicular to the *radius vector* at this point, the body cannot move in a circle.

(92.) *To determine the angular velocity.*—Since $h=\sqrt{\mu a(1-e^2)}$, we have

In the ellipse, $a=\frac{\sqrt{\mu a(1-e^2)}}{r^2}$.

In the hyperbola, $a=\frac{\sqrt{\mu a(e^2-1)}}{r^2}$.

In the parabola, $a=\frac{\sqrt{2D.\mu}}{r^2}$.

The squares of the angular velocities in the curve and equidistant circle, are in the same ratio as the centrifugal and centripetal forces.—*See Art.* (79.)

(93.) The angular velocities about the two foci of the ellipse may be compared as follows:

$$r=\frac{a.(1-e^2)}{1+e\cos.\omega},\qquad r'=\frac{a.(1-e^2)}{1-e\cos.\omega'},$$

$$\frac{dr}{d\omega}=\frac{r^2.e\sin.\omega}{a.(1-e^2)},\qquad \frac{dr'}{d\omega'}=-\frac{r'^2.e\sin.\omega'}{a.(1-e^2)};$$

but $r+r'=2a$, and therefore $dr=-dr'$: and also, $r.\sin.\omega=r'.\sin.\omega'$. Hence $\frac{d\omega}{d\omega'}=\frac{r'}{r}$, or, $\frac{a}{\mathrm{A}}=\frac{r'}{r}$, where A denotes the angular velocity about the empty focus. Now $a=\frac{h}{r^2}$, therefore, $\mathrm{A}=\frac{h}{rr'}=\frac{h}{b'^2}$. From this it appears that the angular velocity about the empty focus is more nearly uniform than about the centre of force, as b'^2 is less subject to variation than r^2.—*Newt. Princ. Lib.* 3. *Prop.* 17.

(94.) *To find the time of describing any portion of the elliptic orbit.*—We have

$$dt=\frac{r^2d\omega}{h}=\frac{a.(1-e^2)}{\sqrt{\mu a(1-e^2)}}.\frac{dr}{e.\sin.\omega}.$$

but

$$1+\cos.\omega=\frac{ae-(r-a)}{re}.(1-e).$$

and

$$1-\cos.\omega=\frac{ae+(r-a)}{re}.(1+e).$$

Hence

$$\sin.\omega=\frac{\sqrt{a^2e^2-(r-a)^2}}{re}.(1-e^2).$$

Using this value for sin. ω in the expression for dt, we obtain

$$dt=\sqrt{\frac{a}{\mu}}.\frac{rdr}{\sqrt{a^2e^2-(a-r)^2}}.$$

In order to integrate this, assume $\sqrt{a^2e^2-(a-r)^2}=ae.\sin.u$.

Then $(a-r)=ae.\cos.u$, and $dr=ae.\sin.u.du$. Introducing these values in the expression for dt, we have

$$dt=\sqrt{\frac{a}{\mu}}\cdot(a-ae.\cos.u)\,du;$$

therefore, integrating, we obtain

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.(u-e.\sin.u).$$

As we introduce no constant, it is the same thing as if we considered t and u to commence together, and that therefore we count the time from the nearer apside. The angle u thus introduced by integration is called the eccentric anomaly.

(95.) Let $n=\sqrt{\frac{\mu}{a^3}}$, the quantity nt is called the *mean anomaly*, being proportional to the time. The angle ω of the orbit is called the true anomaly; to discover an equation between it and u we proceed as follows. By the above it appears that $r=a.(1-e\cos.u)$; and by the equation of the ellipse $r=\frac{a.(1-e^2)}{1+e\cos.\omega}$. Hence we have

$$(1-e\cos.u).(1+e\cos.\omega)=1-e^2;$$

whence

$$\cos.\omega=\frac{\cos.u-e}{1-e\cos.u};$$

but

$$\tan.\frac{\omega}{2}=\sqrt{\frac{1-\cos.\omega}{1+\cos.\omega}}=\sqrt{\frac{(1+e).(1-\cos.u)}{(1-e).(1+\cos.u)}}=\sqrt{\frac{1+e}{1-e}}\cdot\tan.\frac{u}{2}.$$

For a complete discussion of these equations, *vide* Trig. p. 161.

(96.) An approximate value for ω in terms of nt may be obtained. Neglecting the square and higher powers of e the equation of the elliptic orbit gives us

$$r^2 = a^2.(1-2\,e.\cos.\,\omega).$$

Substituting this value in the equation $cdt = \sqrt{\mu a.(1-e^2)}.dt = r^2 d\omega$, we shall have, neglecting e^2, and writing n for $\frac{\sqrt{\mu}}{a^{\frac{3}{2}}}$,

$$n\,dt = (1-2\,e.\cos.\,\omega).\,d\omega.$$

Integrating this, and rejecting the constant as we count t from the perihelion, we have

$$nt = \omega - 2\,\varepsilon.\sin.\,\omega,$$

whence

$$\omega = nt + 2\,\varepsilon.\sin.\,\omega. \qquad (a)$$

This equation affords us a means of approximating at pleasure to the value of ω. If $e=0$, $\omega=nt$; using this value of ω in the second member of the equation, we have as a first approximation $\omega = nt + 2\,\varepsilon.\sin.\,nt$. Using this more correct value of ω in the second member of equation (a), we obtain a second approximation, &c.

The corresponding value of r is $a.(1-e.\cos.\,nt)$, by means of which, and equation (a), we can nearly determine a planet's place in its orbit for any instant.

(97.) *To obtain the time in the hyperbola.*—We proceed as follows: $1+\cos.\,\omega = \frac{(a+r)+ae}{re}\cdot(e-1)$, and $1-\cos.\,\omega = \frac{(a+r)-ae}{re}\cdot(e+1)$; multiplying these values, and substituting, we obtain

$$dt = \sqrt{\frac{a}{\mu}}\cdot\frac{rdr}{\sqrt{(a+r)^2-a^2e^2}}.$$

To integrate this, assume $\sqrt{(a+r)^2-a^2e^2}=ae.\tan.\theta$, then

$$(a+r)=ae.\sec.\theta, \qquad \text{and} \quad dr=\frac{ae.\sin.\theta d\theta}{\cos.^2\theta}.$$

Collecting all these values into the expression for dt, we have

$$dt=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\cdot(e.\sec.\theta-1).\frac{d\theta}{\cos.\theta}.$$

Therefore,

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\cdot\left\{e.\int\frac{d\theta}{\cos.^2\theta}-\int\frac{d\theta}{\cos.\theta}\right\}.$$

The first of these integrals is obviously $e.\tan.\theta$, and the second, (*vide Trig.* p. 172.), is $\log.\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$. Hence

$$t=\frac{a^{\frac{3}{2}}}{\sqrt{\mu}}\cdot\left\{e.\tan.\theta-\log.\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)\right\}.$$

The value of θ in this is analogous to that of u in the preceding. We add no constant, as we suppose the time to commence from the perihelion.

(98.) To find an equation between θ and ω, we have

$$r=a.(e.\sec.\theta-1)=\frac{a.(e^2-1)}{e.\cos.\omega+1}.$$

From this it is easy to deduce that $\tan.\frac{1}{2}\omega=\sqrt{\frac{e+1}{e-1}}\cdot\tan.\frac{1}{2}\theta$.

(99.) *In the parabola.*—D denoting the perihelion distance, the equation to the curve is

$$r=\frac{2D}{1+\cos.\omega}=\frac{D}{\cos.^2\frac{\omega}{2}} \qquad \text{and} \quad h=\sqrt{2\mu D}.$$

Hence

$$dt = \frac{r^2 d\omega}{h} = \frac{D^{\frac{3}{2}}.\sqrt{2}}{\sqrt{\mu}}.d.\frac{\omega}{2}.\frac{1}{\cos.^4\frac{\omega}{2}} =$$

$$\sqrt{\frac{2}{\mu}}.D^{\frac{3}{2}}\left\{\frac{d.\frac{1}{2}\omega}{\cos.^2\frac{1}{2}\omega} + \frac{\sin.^2\frac{1}{2}\omega.d.\frac{1}{2}\omega}{\cos.^4\frac{1}{2}\omega}\right\}.$$

Integrating this, the result is

$$t = \sqrt{\frac{2}{\mu}}.D^{\frac{3}{2}}.\left\{\tan.\frac{\omega}{2} + \tfrac{1}{3}.\tan.^3\tfrac{1}{2}\omega\right\}$$

(100.) These values of t can be expressed in terms of r, as follows:—

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.\left\{\cos.^{-1}\frac{a-r}{ae} = \frac{\sqrt{a^2e^2-(a-r)^2}}{a}\right\}.$$

$$t = \frac{a^{\frac{3}{2}}}{\sqrt{\mu}}.\left\{\frac{\sqrt{(a+r)^2-a^2e^2}}{ae} - \log.\phi(r)\right\}.$$

$$t = \tfrac{1}{3}.\sqrt{\frac{2}{\mu}}.(2D+r).\sqrt{r-D}.$$

In the second of these expressions $\phi(r)$ denotes the value, in terms of r, of $\tan.\left(\frac{\pi}{4}+\frac{\theta}{2}\right)$. The third is deduced from the equation of the curve, for it thence appears that $\tan.\frac{\omega}{2} = \sqrt{\frac{r-D}{D}}$.

(101.) *To determine the space fallen through externally in the focal conic sections.*—In the ellipse, by formula (22), we have, (since $n=2$, and $C = \frac{2r.(2a-r)}{a}$),

$$\frac{2}{\Delta} = \frac{2}{r} - \frac{2a-r}{ar}, \qquad \text{therefore, } \Delta = 2a.$$

Hence describe a circle about the focus with the radius $2a$, and a body falling from its circumference towards the centre of force will have when it arrives at the ellipse the velocity therein.

In the parabola, $C = 4r$, and therefore, $\Delta = \infty$.

In the hyperbola, $C = \frac{2r(2a+r)}{a}$; and therefore $\Delta = -2a$. In this case, if we conceive the body to commence from an infinite distance, with the velocity it would have at an infinite distance if it had moved from the distance $2a$ to infinity by a repelling force, when it reaches the curve it shall have the velocity therein.

Hence in the ellipse the velocity is always less than would be acquired by falling from an infinite distance; in the parabola equal to that from infinity; and in the hyperbola greater.

(102.) *To determine how far a body shall fall internally to acquire the velocity in the focal conic sections.*—In the ellipse, $\frac{2}{\Delta'} = \frac{2}{r} + \frac{2a-r}{ar}$, therefore,

$$\Delta' = \frac{2ar}{4a-r}.$$

To determine the locus we can substitute for r its value $\frac{a.(1-e^2)}{1+e\cos.\omega}$, and the polar equation of the locus is

$$\Delta' = \frac{2a.(1-e^2)}{(3+e^2)+4e\cos.\omega}.$$

This is the equation of an ellipse having the same focus as the given one, the eccentricity and axis major of which can be determined from the value of Δ'.

This locus is always an ellipse, since $e' = \frac{4e}{3+e^2} < 1$.

In the parabola, $\Delta' = \frac{r}{2}$; whence the locus is a parabola having the same focus as the given one.

In the hyperbola, $\Delta' = \frac{2ar}{4a+r}$. Hence the locus is an hyperbola having the same focus as the given one.

(103.) *To determine the rate of increase of the linear velocity in the focal conic sections.*—By the formula in Art. (73.), since $\frac{h^2 dp}{p^3 dr} = \pm\frac{\mu}{r^2}$ and $\frac{p}{r} = \frac{b}{b'}$, we have

$$d.v = \pm\frac{\mu}{r^2}\left(1 - \frac{b^2}{b'^2}\right)^{\frac{1}{2}}$$

in the ellipse and hyperbola. To determine where it increases fastest, we substitute for b'^2 its value $r.(2a-r)$, and differentiate the result. We thus arrive at the following cubic equation:

$$r^3 - 4ar^2 + \frac{8a^2+3b^2}{2} \cdot r - \frac{5b^2 a}{2} = 0.$$

The roots of this are the required values of r.

In the parabola, we have $d.v = \frac{\mu}{r^2} \cdot \left(\frac{r-D}{r}\right)^{\frac{1}{2}}$. To determine when it is a maximum, we have $r = 5r - 5D$, by differentiating the value of $d.v$. Hence

$$r = \tfrac{5}{4}D.$$

(104.) *To determine the rate of increase of the angular velocity in the focal conic sections.*—In the parabola $p^2 = r.D$, hence

$$\frac{dp}{dr} = \tfrac{1}{2}\sqrt{\frac{D}{r}},$$

and consequently we have, by formula (25),

$$da = \frac{h^2}{2} \cdot \sqrt{\frac{D}{r}} \cdot \frac{1}{r^3 D} \cdot \sqrt{1 - \frac{D}{r}} \cdot dt.$$

When the angular velocity increases fastest, it appears from this that $r = \frac{8\,D}{7}$.

To investigate the same in the ellipse and hyperbola, we shall follow a different method from that suggested by formula (25).

By form. (19), $a = \frac{h}{r^2}$, whence

$$da = \frac{-2h\,dr}{r^3}; \qquad \text{but } \frac{dr}{r^2 d\omega} = \frac{\sqrt{a^2e^2-(r-a)^2}}{ra.\sqrt{1-e^2}},$$

and $r^2 d\omega$ being equal to hdt, we have consequently da proportional to $\frac{\sqrt{a^2e^2-(r-a)^2}}{r^4}$. When the angular velocity increases fastest, it appears from this that $r = \frac{7a \pm a.\sqrt{1+48e^2}}{6}$.

PROP. XXXII.

(105.) *To investigate the law of force, velocity, &c. in the conic sections, the centre being the centre of force.*

By the polar equation of the curve, the centre being the pole, we have

$$r^2 = \frac{a^2.(1-e^2)}{1-e^2\cos.^2\omega},$$

whence

$$\frac{1}{r^2} = \frac{1}{a^2(1-e^2)} - \frac{e^2.\cos.^2\omega}{a^2(1-e^2)}.$$

Differentiating this, squaring, and dividing by r^4, we have

$$\frac{dr^2}{r^4d\omega^2}=\frac{e^2.\cos.^2\omega}{a^2.(1-e^2)}\cdot\frac{e^2.\sin.^2\omega}{a^2(1-e^2)}.r^2.$$

Eliminating by the equation of the curve, we have

$$\frac{dr^2}{r^4d\omega^2}=\left\{\frac{1}{a^2(1-e^2)}-\frac{1}{r^2}\right\}\cdot\left\{\frac{1}{r^2}-\frac{1}{a^2}\right\}.r^2.$$

and from this, by differentiating,

$$d.\frac{dr^2}{2r^4d\omega^2}=\frac{dr}{r^3}-\frac{rdr}{a^4.(1-e^2)}.$$

Introducing this value in formula (14), we have

$$F=\frac{h^2.r}{a^4.(1-e^2)}.$$

In the hyperbola the force is repelling, or $F=-\frac{h^2.r}{a^4.(e^2-1)}$.

In general, let $\frac{h^2}{a^4(1-e^2)}=\mu$, and we have in the central conic sections $F=\pm\mu r$; and therefore F varies directly as the distance from the centre of force.

This expression may easily be determined by any of the other formulæ for the force given in the preceding section. For instance, formula (8) may be used for the purpose as follows. In the central conic sections $\frac{1}{p^2}=\frac{b'^2}{a^2b^2}=\frac{a^2+b^2-r^2}{a^2b^2}$, and therefore

$$\frac{dp}{p^3dr}=\frac{r}{a^2b^2},$$

from which

$$F=\frac{h^2.r}{a^2b^2}=\frac{h^2r}{a^4(1-e^2)},$$

the same as above.

(106.) *To compare the centripetal and centrifugal forces.*—The chord of curvature through the centre of the ellipse or hyperbola is the parameter of the point, and is therefore equal to $\frac{2b'^2}{r}$, and $\sin.^2\theta = \frac{a^2b^2}{r^2b'^2}$. Hence from formula (15), we have

$$\frac{\text{F}}{\phi} = \frac{r^4}{a^2b^2}.$$

(107.) *To determine the paracentric velocity.*—From formula (16), and the above investigation of the force, we have

$$(\text{P}.v)^2 = \frac{\mu . r^2 . e^4 . \sin.^2\omega . \cos.^2\omega}{b^2}.$$

From this and the equation of the curve the *max.* and *min.* values may easily be determined.

(108.) *To determine the velocity.*—Putting for F and C their values in form. (17), we have

$$v = b' . \sqrt{\mu}.$$

And by form. (18),

$$\frac{v}{v'} = \frac{b'}{r}.$$

(109.) *To compute the angular velocity.*—By form. (19), we have

$$\alpha = \frac{ab\sqrt{\mu}}{r^2}. \qquad \text{and } \frac{\alpha}{\alpha'} = \frac{a^2b^2}{r^4}.$$

The quantities α and α', as also F and ϕ, are equal when $r = \sqrt{ab}$.

(110.) *To determine the time.*—We have

$$dt = \frac{r^2 d\omega}{h}$$

by form. (21).

Introducing rectangular for polar co-ordinates, we have

$$dt=\frac{y.dx-xdy}{a^2.\sqrt{\mu.(1-e^2)}}.$$

The equation of the ellipse referred to its centre is $a^2y^2+b^2x^2=a^2b^2$. By this we find

$$ydx-xdy=\frac{ab.dx}{\sqrt{a^2-x^2}}.$$

Therefore,

$$dt=\frac{dx}{\sqrt{\mu}.\sqrt{a^2-x^2}},\quad \text{since } a.\sqrt{1-e^2}=b.$$

Integrating, we have

$$t=\frac{1}{\sqrt{\mu}}\cdot\sin.^{-1}\frac{x}{a}.$$

We add no constant, as we consider t and x to commence together, or in other words, count the time from the extremity of the axis minor.

From this it appears that if several concentric ellipses have the same axis major, the locus of the points arrived at in a given time after leaving the extremities of the minor axes, is a right line parallel to the minor axis.

For the whole revolution $t=\frac{2\pi}{\sqrt{\mu}}$, which is therefore the same for all ellipses about the same centre. And about different centres the periodic times are inversely as the square roots of the absolute forces.

(111.) In the hyperbola we have $dt=-\frac{dx}{\sqrt{\mu}.\sqrt{x^2-a^2}}$. In this substitute $x-z$ for $\sqrt{x^2-a^2}$, and we have

$$dt=\frac{dz}{\sqrt{\mu}.z}.$$

Hence, integrating and restoring its value for z, we find

$$t=\frac{1}{\sqrt{\mu}}\cdot l\left\{\frac{x-\sqrt{x^2-a^2}}{a}\right\}$$

the time being supposed to commence when $x=a$.

Passing from logarithms to numbers, we find x in terms of t. We have then

$$x=\frac{a}{2}\cdot\left(e^{t\sqrt{\mu}}+e^{-t\sqrt{\mu}}\right).$$

Hence in several hyperbolæ, with the same axis major, the locus of the points arrived at in a given time is a right line parallel to the axis minor.

(112.) *We have for the space fallen through externally.*—In the ellipse $\Delta^2=a^2+b^2$, and in the hyperbola $\Delta^2=a^2-b^2$. This follows instantly from formula (22), by writing (-1) for n, and for C its proper value; and using the negative sign in the ellipse where the force is attractive, and the positive in the hyperbola where it is repulsive.

The *locus* of the points is in both cases a circle, which becomes a point in the equilateral hyperbola, and imaginary if $b > a$.

(113.) *For the space fallen through internally.*—If we substitute in form. (23) for n and C their values, we readily obtain in the ellipse $\Delta'^2=r^2-b'^2$, and in the hyperbola $\Delta'^2=r^2+b'^2$. The locus is in both cases the same as the locus of the intersections of the parameters with their respective diameters, which is a species of Lemniscata.

(114.) *To determine the increment of the linear velocity.*—From form. (24) we can easily obtain

$$d.v=\mu r.\sqrt{1-\frac{a^2b^2}{r^2.b'^2}}.$$

This being differentiated, and the result equated to cypher, the result for the *max.* value is

$$r=\sqrt{(a-b)^2+ab}.$$

(115.) *For the increment of the angular velocity.*—By form. (25.) we have

$$da=\frac{\mu.p}{r}.\sqrt{1-\frac{p^2}{r^2}}.$$

This quantity is obviously greatest when $p=\frac{r}{\sqrt{2}}$; or if the angle under the equal conjugate diameters be equal to, or greater than 45°, it is greatest at their extremities.

PROP. XXXIII.

(116.) *To investigate the law of force in the curve whose equation is* $r=\frac{a.(1-e^2)}{1+e\cos.m\omega}$.

By differentiating and squaring,

$$\frac{dr^2}{r^4d\omega^2}=\frac{m^2e^2.\sin.^2m\omega}{a^2.(1-e^2)^2}.$$

Hence

$$d.\frac{dr^2}{2r^4d\omega^2}=\frac{m^3.e^2.\sin.m\omega.\cos.m\omega}{a^2.(1-e^2)}=\frac{m^2dr}{r^2}.\left\{\frac{1}{r}-\frac{1}{a.(1-e^2)}\right\}.$$

Substituting this value in formula (14), we have

$$\mathrm{F}=h^2.\left\{\frac{1-m^2}{r^3}+\frac{m^2}{a.(1-e^2)}.\frac{1}{r^2}\right\}.$$

This is the equation of the moveable ellipse and the law of force therein, as it is given in the ninth section of the Principia.

The other conditions of the motion, such as the velocity, time, &c. may be determined as in the preceding examples.

PROP. XXXIV.

(117.) *To investigate the law of force in the spiral of Archimedes.*

The equation of this spiral is $r = a\omega$. Differentiating it and squaring, we have

$$\frac{dr^2}{r^4 d\omega^2} = \frac{a^2}{r^4},$$

and therefore,

$$d.\frac{dr^2}{2r^4 d\omega^2} = \frac{2a^2 dr}{r^5}.$$

Introducing this expression in formula (14), we have

$$F = h^2.\left\{\frac{1}{r^3} - \frac{2a^2}{r^5}\right\}.$$

The two last propositions give us instances of the force varying as two powers of the distance.

It would be easy to enlarge upon the subject of this section, but the cases that have been given may suffice to exemplify the formulæ of Sect. III., as applied to the direct problem of central forces. Our next object shall be to exemplify them on a more enlarged scale in their application to the inverse method, which is incomparably the most important part of our subject, as it directly leads to the determination of the true motions of the bodies of our system, and of their places for any assigned epoch.

SECTION V.

ON THE INVESTIGATION OF THE ORBITS DESCRIBED BY BODIES, WHEN THE LAWS OF FORCE AND THE INITIAL CONDITIONS OF THE MOTION ARE ASSIGNED.

(118.) We shall next proceed to the inverse method of central forces, or the discovery of orbits from given laws of force. This section we shall divide as the preceding one into the cases of a force acting along a system of parallels, or in other words, tending to an infinitely distant centre; and of a force tending to a given centre, and varying as some function of the distance therefrom.

§ I.

PROP. XXXV.

(119.) *When the force is constant, and acts along a system of parallel lines, to investigate the nature of the orbit described by a material point projected obliquely to the direction of the force.*

Assuming as axes of co-ordinates two right lines through the point of departure, one coinciding with the direction of the force, and the other perpendicular thereto, we have then

$$\frac{d^2y}{dt^2}=-g, \qquad \frac{d^2x}{dt^2}=0.$$

The first integrals of these equations are

$$\frac{dy}{dt}=-gt+b, \qquad \frac{dx}{dt}=c. \qquad (a)$$

Integrating a second time, we obtain

$$y=-g.\frac{t^2}{2}+bt, \qquad x=ct.$$

We introduce no constants in this second integration, since, having selected the origin at the point of departure, x, y, and t vanish together. Eliminating t we obtain

$$y=\frac{b}{c}.x-\frac{g.x^2}{2c^2}.$$

(120.) This is the equation of a parabola with its axis coinciding with the direction of the force, which is therefore the required orbit. Thus we see how the *nature* of the orbit is determined merely by the *law* and the *absolute value* of the acting force. To determine its *form* it is necessary to assign values to the constants b and c.

Since $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are the components of the velocities along the axes of y and x respectively, it appears by equations (a) that b is equal to the former component at the origin of the motion, and that c is always equal to the latter. Let the velocity of projection be a, and θ the angle which its direction makes with the axis of x, then

$$b=a.\sin.\theta, \qquad c=a.\cos.\theta,$$

and therefore the equation of the curve becomes

$$y=x.\tan.\theta-\frac{g.x^2}{2\,a^2\cos.^2\theta}.$$

(121.) The force of gravity at the surface of the earth, if we abstract the air's resistance, is a force of the description mentioned in this proposition, as the lines along which it acts converging to the earth's centre may be considered parallel without any sensible error.

(122.) The equation of a projectile at the earth's surface may also be arrived at as follows:—assuming an horizontal line as axis of x, and a vertical one as axis of y, and the point of departure as origin ; if the force of gravity did not act, we should have $y=x.\tan.\theta$, but the action of gravity diminishes this value of y in the time t, by the quantity $\frac{g.t^2}{2}$. Hence we have for the equation of the orbit

$$y=x.\tan.\theta-\frac{g.t^2}{2}. \qquad (1)$$

Since the horizontal component of the velocity is uniform, we have $x=t.a.\cos.\theta$. whence the equation may be written

$$y=x.\tan.\theta-\frac{g.x^2}{2a^2\cos.^2\theta}.$$

Or denoting by h the space through which the velocity a would be acquired, we have

$$y=x.\tan.\theta-\frac{x^2}{4h.\cos.^2\theta}. \qquad (2)$$

(123.) By making $y=0$ in this equation, we obtain the horizontal range $x=2h.\sin.2\theta$; and therefore denoting the co-ordinates of the vertex by x', y', we obtain

$$x'=h.\sin.2\theta, \qquad \text{and } y'=h.\sin.^2\theta.$$

Since in the parabola the tangent bisects the angle under the focal *radius vector* and diameter, it will readily appear by constructing the figure that the distance from the origin to the focus is equal to h. Hence, denoting the co-ordinates of the focus by x'', y'', we shall have

$$x''=h.\sin.2\theta, \qquad y''=h.\cos.2\theta.$$

(124.) Let ϕ be the angle at which a right line in the plane of the projectile is inclined to the horizon, and b the distance along the axis of y, of its intersection ; then the equation of this right line is

$$y=x.\tan.\phi+b.$$

To determine the intersection of the path of the projectile with this line we need but equate the values of y in both, and solve for x. We thus obtain the values of x and y as follows:

$$\left.\begin{aligned} x&=2h.\frac{\cos.\theta}{\cos.\phi}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\phi}\right\} \\ y&=b+2h.\frac{\sin.\phi.\cos.\theta}{\cos.^2\phi}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\phi}\right\}\end{aligned}\right\}(3)$$

The time may be found by dividing the value of x, thus obtained, by the horizontal component of the velocity. This gives us

$$t=\sqrt{\frac{2h}{g}}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\theta}\right\} \quad (4)$$

(125.) *To determine the direction of projection that the abscissa of the point, where the projectile meets a given line, may be a maximum.*—Equating the values of y we have

$$x.\tan.\phi+b=x.\tan.\theta-\frac{x^2}{4h.\cos.^2\theta}. \quad (b)$$

In this equation it is required to determine θ, so that x be a *maximum*. For this purpose let us differentiate, considering $dx=0$, and we obtain thus

$$x=\frac{2h}{\tan.\theta}.$$

Substituting this value for x in equation (b), we obtain

$$\tan.\theta=\frac{h.\tan.\phi\pm\sqrt{h^2.\sec.^2\phi-hb}}{h-b}. \quad (5$$

If $b=0$, $\tan.\theta=\tan.\phi\pm\sec.\phi=\tan.\frac{1}{2}(\phi\pm90)$.—*Trig.* p. 14. Hence the direction which produces the greatest

SECTION V.

ON THE INVESTIGATION OF THE ORBITS DESCRIBED BY BODIES, WHEN THE LAWS OF FORCE AND THE INITIAL CONDITIONS OF THE MOTION ARE ASSIGNED.

(118.) WE shall next proceed to the inverse method of central forces, or the discovery of orbits from given laws of force. This section we shall divide as the preceding one into the cases of a force acting along a system of parallels, or in other words, tending to an infinitely distant centre; and of a force tending to a given centre, and varying as some function of the distance therefrom.

§ I.

PROP. XXXV.

(119.) *When the force is constant, and acts along a system of parallel lines, to investigate the nature of the orbit described by a material point projected obliquely to the direction of the force.*

Assuming as axes of co-ordinates two right lines through the point of departure, one coinciding with the direction of the force, and the other perpendicular thereto, we have then

$$\frac{d^2y}{dt^2}=-g, \qquad \frac{d^2x}{dt^2}=0.$$

The first integrals of these equations are

$$\frac{dy}{dt}=-gt+b, \qquad \frac{dx}{dt}=c. \qquad (a)$$

Integrating a second time, we obtain

$$y = -g.\frac{t^2}{2}+bt, \qquad x=ct.$$

We introduce no constants in this second integration, since, having selected the origin at the point of departure, x, y, and t vanish together. Eliminating t we obtain

$$y=\frac{b}{c}\cdot x-\frac{g.x^2}{2c^2}.$$

(120.) This is the equation of a parabola with its axis coinciding with the direction of the force, which is therefore the required orbit. Thus we see how the *nature* of the orbit is determined merely by the *law* and the *absolute value* of the acting force. To determine its *form* it is necessary to assign values to the constants b and c.

Since $\frac{dy}{dt}$ and $\frac{dx}{dt}$ are the components of the velocities along the axes of y and x respectively, it appears by equations (a) that b is equal to the former component at the origin of the motion, and that c is always equal to the latter. Let the velocity of projection be a, and θ the angle which its direction makes with the axis of x, then

$$b=a.\sin.\theta, \qquad c=a.\cos.\theta,$$

and therefore the equation of the curve becomes

$$y=x.\tan.\theta-\frac{g.x^2}{2a^2\cos.^2\theta}.$$

(121.) The force of gravity at the surface of the earth, if we abstract the air's resistance, is a force of the description mentioned in this proposition, as the lines along which it acts converging to the earth's centre may be considered parallel without any sensible error.

(122.) The equation of a projectile at the earth's surface may also be arrived at as follows:—assuming an horizontal line as axis of x, and a vertical one as axis of y, and the point of departure as origin; if the force of gravity did not act, we should have $y=x.\tan.\theta$, but the action of gravity diminishes this value of y in the time t, by the quantity $\frac{g.t^2}{2}$. Hence we have for the equation of the orbit

$$y=x.\tan.\theta-\frac{g.t^2}{2}. \qquad (1)$$

Since the horizontal component of the velocity is uniform, we have $x=t.a.\cos.\theta$. whence the equation may be written

$$y=x.\tan.\theta-\frac{g.x^2}{2a^2\cos.^2\theta}.$$

Or denoting by h the space through which the velocity a would be acquired, we have

$$y=x.\tan.\theta-\frac{x^2}{4h.\cos.^2\theta}. \qquad (2)$$

(123.) By making $y=0$ in this equation, we obtain the horizontal range $x=2h.\sin.2$[illegible] and therefore denoting the co-ordinates of the vertex by x', y', we obtain

$$x'=h.\sin.2\theta, \qquad \text{and } y'=h.\sin.^2\theta.$$

Since in the parabola the tangent bisects the angle under the focal *radius vector* and diameter, it will readily appear by constructing the figure that the distance from the origin to the focus is equal to h. Hence, denoting the co-ordinates of the focus by x'', y'', we shall have

$$x''=h.\sin.2\theta, \qquad y''=h.\cos.2\theta.$$

(124.) Let ϕ be the angle at which a right line in the plane of the projectile is inclined to the horizon, and b the distance along the axis of y, of its intersection; then the equation of this right line is

$$y=x.\tan.\phi+b.$$

To determine the intersection of the path of the projectile with this line we need but equate the values of y in both, and solve for x. We thus obtain the values of x and y as follows:

$$\left.\begin{aligned} x&=2h.\frac{\cos.\theta}{\cos.\phi}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\phi}\right\} \\ y&=b+2h.\frac{\sin.\phi.\cos.\theta}{\cos.^2\phi}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\phi}\right\} \end{aligned}\right\}(3)$$

The time may be found by dividing the value of x, thus obtained, by the horizontal component of the velocity. This gives us

$$t=\sqrt{\frac{2h}{g}}.\left\{\sin.(\theta-\phi)\pm\sqrt{\sin.^2(\theta-\phi)-\frac{b}{h}.\cos.^2\theta}\right\} \quad (4)$$

(125.) *To determine the direction of projection that the abscissa of the point, where the projectile meets a given line, may be a maximum.*—Equating the values of y we have

$$x.\tan.\phi+b=x.\tan.\theta-\frac{x^2}{4h.\cos.^2\theta}. \quad (b)$$

In this equation it is required to determine θ, so that x be a *maximum*. For this purpose let us differentiate, considering $dx=0$, and we obtain thus

$$x=\frac{2h}{\tan.\theta}.$$

Substituting this value for x in equation (b), we obtain

$$\tan.\theta=\frac{h.\tan.\phi\pm\sqrt{h^2.\sec.^2\phi-hb}}{h-b}. \quad (5$$

If $b=0$, $\tan.\theta=\tan.\phi\pm\sec.\phi=\tan.\frac{1}{2}(\phi\pm90)$.—*Trig.* p. 14. Hence the direction which produces the greatest range* is that which bisects the angle under the line from the origin, and the axis of y. And if the range be horizontal, the angle of elevation must be 45°.

(126.) *To determine the value of the maximum range.*—The range along the plane is equal to the abscissa of the point where the projectile meets the plane divided by the cosine of the angle of the plane's elevation. But the abscissa answering to the *max.* range is $\frac{2h}{\tan.\phi+\sec.\phi}$, therefore, if R denote the *max.* range, we have

$$R=\frac{2h}{1+\sin.\phi}. \tag{6}$$

(127.) *To determine the range along a plane through the origin, and the time of describing it.*—We have for this $b=0$ in equations (3), and $x=R.\cos.\phi$, hence

$$R=\frac{4h.\sin.(\theta-\phi).\cos.\theta}{\cos.^2\phi}. \tag{7}$$

And by equation (4),

$$t=\sqrt{\frac{2h}{g}}.\frac{2\sin.(\theta-\phi)}{\cos.\phi}=\frac{R.\cos.\phi}{\sqrt{2gh}.\cos.\theta}. \tag{8}$$

Let us suppose another elevation, θ', to produce an equal range, then from formula (7) we have

$$\sin.(\theta-\phi).\cos.\theta=\sin.(\theta'-\phi).\cos.\theta',$$

or,

$$\sin.(2\theta-\phi)-\sin.\phi=\sin.(2\theta'-\phi)-\sin.\phi.$$

Hence,

$$2\theta-\phi=\pi-(2\theta'-\phi),$$

and consequently,

$$\theta+\theta'=\frac{\pi}{2}+\phi.$$

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* We mean by the *range* the distance from the origin to where the projectile meets the given line through the origin.

This shews us that if the velocity of projection be given, there are two elevations that give equal ranges, and that the directions of these elevations are equally inclined to that direction which produces a *max.* range.

(128.) *A body being projected from a point above an horizontal plane, to find where it meets it.*—For this purpose it is only necessary to suppose $\phi=0$ in formula (3), and to change the sign of b. This gives us

$$x=2h.\cos.\theta\left\{\sin.\theta\pm\sqrt{\sin.^2\theta+\frac{b}{h}}\right\}$$

and

$$t=\sqrt{\frac{2h}{g}}.\left\{\sin.\theta\pm\sqrt{\sin.^2\theta+\frac{b}{h}}\right\}.$$

To determine the direction of projection that will make x a *max.* in this case, we make similar changes in formula (5), whence we obtain

$$\tan.\theta=\sqrt{\frac{h}{h+b}}.$$

Hence the angle which gives the greatest range in this case is less than 45°.

(129.) *A body is projected in a given direction with a given velocity from the summit of a paraboloidal hill, to find where it shall meet it.*—Referring the parabola, which is a section of the hill, to the horizontal and vertical lines as axes of co-ordinates, its equation is $x^2=-py$, where p is its parameter. Equating the value of y from this with the value of y from equation (2), we obtain

$$x=\frac{2ph.\sin.2\theta}{p-4h.\cos.^2\theta}.$$

If $4h.\cos.^2\theta=p$ they never meet, since x is infinite. In this case the two parabolæ have equal parameters, since $4h.\cos.^2\theta$ is the parameter of the projectile's path, as is obvious from Art. (123.)

If $4h. \cos.^2\theta > p$ the curves diverge, and if the path of the projectile be continued backwards it intersects the hill.

In the same way we may find where the projectile meets any curve in the same plane with it, whose equation is given.

(130.) *A body is projected from a given point with a given velocity, to find the direction that it may just touch a given line.*—Let the equation of the given line be $y = x. \tan. \phi + b$. Then since at the point where the projectile touches the plane we must have the co-ordinates common, we have

$$x. \tan. \phi + b = x. \tan. \theta - \frac{x^2}{4h. \cos.^2\theta}.$$

Also, since the curve and the line touch, we must have $\frac{dy}{dx}$ the same for both; hence we have

$$\tan. \phi = \tan. \theta - \frac{x}{2h. \cos.^2\theta}.$$

Eliminating x between these [illegible] equations we can easily deduce

$$\sin. (\theta - \phi) = \pm\sqrt{\frac{b. \sin. \phi. \cos. \phi}{h}} = \pm\sqrt{\frac{b. \sin. 2\phi}{2h}}.$$

It appears from this that there are two directions which answer the required condition.

(131.) *A body is projected from a given point with a given velocity, required the direction that it may strike a given line perpendicularly.*—In this case $\left(\frac{dy}{dx}\right)$ of the curve is equal to $\left(-\frac{dx}{dy}\right)$ of the line, hence we have

$$-\cot. \phi = \tan. \theta - \frac{x}{2h. \cos.^2\theta},$$

and therefore

$$x=\frac{2h.\cos.\theta.\cos.(\theta-\phi)}{\sin.\phi}.$$

Equating the values of y, and substituting this value for x, we have

$$\frac{2h.\cos.\theta.\cos.(\theta-\phi)}{\cos.\phi}+b=\frac{2.h.\sin.\theta.\cos.(\theta-\phi)}{\sin.\phi}-\frac{h.\cos.^2(\theta-\phi)}{\sin.^2\phi},$$

from which may readily be deduced the following,

$$\cos.2.(\theta-\phi)-2.\tan.\phi.\sin.2.(\theta-\phi)=-\frac{h+b.\tan.\phi.\sin.2\phi}{h}$$

and from this $(\theta-\phi)$ may be found

(132.) *Given the direction of projection, it is required to find the velocity that the projectile may pass through a given point.*—Let α and β be the co-ordinates of the given point. Substituting these for x and y in equation (2), and solving for a^2, the result obtain[illegible]

$$a^2=\frac{g\alpha^2}{2\cos.^2\theta.(\alpha.\tan.\theta-\beta)}.$$

(133.) *To find the direction, given the velocity of projection, that the projectile may pass through a given point whose co-ordinates are α and β.*—Substitute α and β for x and y respectively in equation (2), and solve for $\tan.\theta$. The result obtained is

$$\tan.\theta=\frac{2h\pm\sqrt{4h^2-4h\beta-\alpha^2}}{\alpha}.$$

(134.) *To find the locus of the vertex when the direction of projection is given.*—Eliminating h from the co-ordinates of the vertex obtained in Art. (123), we obtain

$$\frac{y'}{x'}=\tfrac{1}{2}.\tan.\theta;$$

and therefore the locus required is a right line through the origin.

(135.) *To find the locus of the vertex when the velocity of projection is given.*—Squaring the value of x', we have

$$x'^2 = 4h^2 . \sin.^2\theta \, (1 - \sin.^2\theta),$$

but

$$\sin.^2\theta = \frac{y'}{h};$$

therefore,

$$x'^2 = 4hy' \left(1 - \frac{y'}{h}\right),$$

and consequently

$$4y'^2 + x'^2 - 4hy' = 0.$$

Hence the locus required is an ellipse with the origin at the lower extremity of the axis minor. The axis major is horizontal and equal to $2h$, and the axis minor is vertical and equal to h.

(136.) *To find the locus of the foci when the direction of projection is given.*—By division we have

$$\frac{y''}{x''} = \cot. 2\theta,$$

and therefore the locus required is a right line through the origin.

(137.) *To find the locus of the foci when the velocity of projection is given.*—Squaring the values of x'' and y'', and adding them, we obtain

$$x''^2 + y''^2 = h^2,$$

whence the locus is a circle.

(138.) *Given the direction of projection, to find the locus of the points arrived at in a given time.*—By equation (1) it appears that the locus is a right line parallel to the direction of projection, and intersecting the axis of y at a distance from the origin $\frac{gt^2}{2}$.

(139.) *Several bodies are projected in the same direction with different velocities, find the path traced out by their common centre of gravity.*—Taking the direction of projection and vertical as axes of co-ordinates, the equations of the several parabolæ are $y^2=px$, $y'^2=p'x$, &c.; let $y_{,}$ be the ordinate of the centre of gravity. Then by the theory of the centre of gravity we have

$$y_{,}=\frac{\Sigma.\,my}{\Sigma.\,m},$$

and therefore,

$$y_{,}^2=\frac{\Sigma.m^2y^2+2\,\Sigma.mm'.yy'}{(\Sigma.m)^2}=\frac{\Sigma.m^2p+2\,\Sigma.mm'.\sqrt{pp'}}{(\Sigma.m)^2}.x=p_{,}.x.$$

Hence the locus required is a parabola whose parameter is $p_{,}$.

(140.) *Given the velocity of projection, find the locus of the points arrived at in a given time.*—Since $x=at.\cos.\theta$, we have

$$\text{c[illegible]}=\frac{x}{at},$$

and consequently

$$\tan.\theta=\frac{\sqrt{a^2t^2-x^2}}{x},$$

substituting this value for $\tan.\theta$ in equation (1), we have

$$y=\sqrt{a^2t^2-x^2}-\frac{g.t^2}{2}.$$

Hence the locus is a circle with its centre on the axis of y, at a distance below the origin equal to $\frac{g.t^2}{2}$, and the radius of which is at.

(141.) *Bodies having run down the chords of a circle whose radius is r, pass into a vacuum, find the locus of the* *points arrived at in a given time.*—The velocity in this case is

$$\sqrt{gr.\sin.\theta},$$

therefore,

$$x=\sqrt{gr}.\cos.\theta.\sin.\theta.t,$$

and by the equation of the curve

$$y=x.\tan.\theta-\frac{g.t^2}{2}.$$

By the latter equation

$$\tan.\theta=\frac{y+m}{x},$$

(m denoting $\frac{gt^2}{2}$); and therefore by the former we have

$$x=\sqrt{gr}.\frac{x.(y+m)}{x^2+(y+m)^2}.t.$$

Hence the equation of the required locus is

$$x^2+y^2+(2m-t\sqrt{\text{[illegible]}}y-m.t\sqrt{gr}=0.$$

This is the equation of a circle whose centre is on the axis of y, at a distance from the origin equal to $-\left(m-\frac{t\sqrt{gr}}{2}\right)$, and the radius of which is $\sqrt{m^2+\frac{1}{4}grt^2}$.

(142.) *Several inclined planes being drawn from the origin of co-ordinates, find the locus of the extremities of the maximum ranges on them.*—The length of a *maximum* range on a plane of given elevation is, by Art. (126.), $\frac{2h}{1+\sin.\phi}$. If the elevation be supposed variable, and R equal to its length, the polar equation of the required locus is $R=\frac{2h}{1+\sin.\phi}$. This is obviously the equation of a parabola referred to its focus, the axis being vertical, and the vertex of the locus at an altitude above the origin equal to h.

This locus is obviously the parabola which would generate a paraboloid to touch all the parabolæ, described by bodies projected from a given point with a given velocity.

(143.) *Two bodies are projected with given velocities and in given directions from the same point, find the locus of their centre of gravity.*—Since gravity acts equally on each of the bodies, they have the same abscissa at the end of the same time, and their ordinates are to each other as the velocities of projection. Moreover, the angle at which these ordinates are respectively inclined is given, being equal to the difference of the angles of elevation. Hence the triangle formed by these ordinates, and the line joining their extremities is given in species, and consequently the locus of the point dividing the base in a given ratio is determined. This locus in the present instance is a parabola.

This proposition contains the full theory of projectiles in *vacuo*, when the force acts along parallel lines. These hypotheses are not strictly conformable to the true case of nature, as the directions of the force converge to the earth's centre, and as the motion takes place in a resisting medium. The error arising from the latter cause is, however, incomparably greater than from the former, so much so that the deviations which take place from the results obtained by theory are frequently equal to half the results themselves. It is, therefore, indispensably necessary in order to arrive at results at all practical to consider the resistance which is afforded by the atmosphere to the motions of projectiles. This we propose to do in a note on this part of the present treatise, to which the reader is referred.

PROP. XXXVI.

(144.) *To investigate the orbit when the force acts along a system of parallel lines, and varies inversely as the cube of the ordinate to a given line.*

For this case the differential equations become

$$\frac{d^2y}{dt^2}=-\frac{\mu}{y^3}, \qquad \frac{d^2x}{dt^2}=0.$$

Integrating the latter, we have

$$x=ct+c'.$$

Let o, Fig. (44), be the point of projection, and mn the line to which the ordinates are applied, then c denotes the component of the velocity at o parallel to mn, and c' shall be equal to cypher if we assume the origin at m, and date the commencement of the motion from o. Hence we have

$$x=ct.$$

Eliminating dt from the former of equations (a), we find

$$\frac{c^2d^2y}{dx^2}=\pm\frac{\mu}{y^3}.$$

Multiplying both sides by $2dy$, and integrating, we obtain

$$\frac{c^2dy^2}{dx^2}=\pm\frac{\mu}{y^2}+\mathrm{C}.$$

Let the line $om=m$, and let the component of the velocity at o, parallel to the axis of y, be b, and we have

$$b^2 = \pm\frac{\mu}{m^2} + \mathrm{C}.$$

Eliminating C, and solving for dx, we find

$$dx = \frac{cmy.dy}{\sqrt{\pm\mu m^2 + (m^2b^2 \mp \mu)y^2}}.$$

Integrating this, and determining the constant from the known value of y when $x=0$, we find

$$x = \frac{cm}{m^2b^2 \mp \mu}.\sqrt{\pm\mu m^2 + (m^2b^2 \mp \mu)y^2} - \frac{cm^3b}{m^2b^2 \mp \mu}.$$

(145.) Take in the figure $mv = \frac{cm^3b}{m^2b^2 \mp \mu}$, (unless $m^2b^2 - \mu = 0$), and change the origin to the point v. Then the equation of the orbit becomes

$$(m^2b^2 \mp \mu).x^2 - c^2m^2y^2 = \frac{c^2m^4\mu}{(m^2b^2 \mp \mu)}.$$

This is the equation of a central conic section, an ellipse if $m^2b^2 \mp \mu < 0.$ and an hyperbola if $> 0.$

(146.) If $m^2b^2 - \mu = 0$, the orbit is a parabola, whose equation is

$$2\sqrt{\mu}.x = cy^2 - cm^2.$$

Take in the figure $mv = -\frac{cm^2}{2\sqrt{\mu}}$, and change the origin to v, and the equation becomes

$$y^2 = \frac{2\sqrt{\mu}}{c}.x.$$

Hence v is the vertex of the parabola.

(147.) If the force be repulsive, or μ positive, the orbit is necessarily an hyperbola.

(148.) If the force be *directly* as the cube of the ordinate, we shall have to integrate the equation

$$dx=\frac{c.\sqrt{2}.dy}{\sqrt{(b-\mu m^4)+\mu y^4}}.$$

where m, b, and c denote the same as in the preceding.

Taking the case when $b=\mu m^4$, the equation becomes, by integration,

$$axy-c.\sqrt{\frac{2}{\mu}}.y+ac\sqrt{\frac{2}{\mu}}=0.$$

This is the equation of an hyperbola with the axis of x, an asymptote.

PROP. XXXVII.

(149.) *To investigate the orbit when the force acts along a system of parallel lines, and varies inversely as the square of the ordinate.*

From the differential equations we can easily obtain for this case

$$\frac{c^2dy^2}{dx^2}=\frac{2\mu}{y}-\frac{2\mu}{m}+b^2,$$

where c, m, and b denote the same as in the preceding examples. Hence we have

$$dx=\frac{c\sqrt{my}.dy}{\sqrt{2m\mu-(2\mu-mb^2).y}}.$$

Multiplying above and below by $\sqrt{y}$, and integrating as in Art. (24.), we find if we denote by n the quantity $\sqrt{\frac{m}{2u-mb^2}}$,

$$x = -cn.\{\sin.^{-1}\sqrt{2\mu n^2 y - y^2} + \sqrt{2\mu n^2 y - y^2}\} + C.$$

If we conceive the origin to be at the point where the curve intersects the axis of x, we may neglect the constant; and then we have for the equation of the curve

$$\sqrt{2\mu n^2 y - y^2} = \sin.\left\{\frac{x}{cn} - \sqrt{2mn^2 y - y^2}\right\}.$$

This is the most general equation of the cycloid.

(150.) If in this case $mb = 2\mu$, the equation of the curve is

$$x = \tfrac{2}{3}.\frac{c.}{\sqrt{2\mu}}\cdot(y^{\frac{3}{2}} - m^{\frac{3}{2}});$$

or taking the origin where the curve cuts the axis of x,

$$x^2 = \frac{4c^2}{18\mu}\cdot y^3.$$

This is the equation of the semi-cubical parabola.

PROP. XXXVIII.

(151.) *To investigate the orbit when the force varies as the ordinate to a given line.*

In this case, proceeding as before, we have

$$\frac{c^2 d^2 y}{dx^2} = -\mu y.$$

Integrating this, and solving for dx, we find

$$dx = \frac{c\,dy}{\sqrt{\mu m^2 + b^2 - \mu y^2}}.$$

Hence

$$x=\frac{c}{\sqrt{\mu}}\cdot\sin.^{-1}\frac{y.\sqrt{\mu}}{\sqrt{\mu m^2-b^2}}+\mathrm{C}.$$

The constant C may be neglected if we take the origin where the curve intersects the axis of x. This gives us

$$y=\sqrt{\frac{\mu m^2-b^2}{\mu}}\cdot\sin.\frac{\sqrt{\mu}}{c}\cdot x.$$

The general equation of the sinusoid.

If the force be repulsive, the equation to be integrated is

$$dx=\frac{c\,dy}{\sqrt{\mu y^2+(b^2-\mu m^2)}}\cdot$$

the integral of which is

$$x=\frac{-c}{\sqrt{\mu}}\cdot\log.\left\{\sqrt{y^2+\frac{b^2-\mu m^2}{\mu}}-y\right\}+\mathrm{C}.$$

If $b^2=\mu m^2$, we have

$$dx=\frac{c\,dy}{\sqrt{\mu}.y},$$

and therefore

$$x=\frac{c}{\sqrt{\mu}}\cdot\log.y.$$

which is the logarithmic curve.

PROP. XXXIX.

(152.) *To determine generally in what cases the equation of the orbit can be determined in finite terms, a point being attracted along a system of parallel lines.*

Let the force be $\frac{\mu}{y^n}$, then, as in the preceding examples,

$$\frac{c^2d^2y}{dx^2}=-\frac{\mu}{y^n}.$$

Multiplying by $2dy$, and integrating, we have

$$\frac{c^2dy^2}{dx^2}=\frac{2\mu}{n-1}\cdot\frac{1}{y^{n-1}}+C.$$

Let $\frac{c^2dy^2}{dx^2}$ be represented by b^2 when $y=m$, then

$$b^2=\frac{2\mu}{n-1}\cdot\frac{1}{m^{n-1}}+C.$$

Hence we have

$$\frac{c^2dy^2}{dx^2}=\frac{2\mu}{n-1}\cdot\left\{\frac{1}{y^{n-1}}-\frac{1}{m^{n-1}}\right\}+b^2=\frac{2\mu}{n-1}\left\{\frac{1}{y^{n-1}}-\frac{1}{\Delta^{n-1}}\right\}.$$

As appears by writing $\frac{2\mu}{n-1}\cdot\frac{1}{\Delta^{n-1}}$ for $\left(\frac{2\mu}{n-1}\cdot\frac{1}{m^{n-1}}-b^2\right)$. Hence the expression to be integrated is

$$dx=c.\sqrt{\frac{n-1}{2\mu}\cdot\Delta^{n-1}}.\frac{y^{\frac{n-1}{2}}.dy}{\sqrt{\Delta^{n-1}-y^{n-1}}}.$$

This formula is precisely of the same form as that given in Art. (32.), p. 106, of this Treatise, and is therefore integrable for the same values of n.

§ II.

(153.) It has been found in Art. (27.) that when the force is directed to a single centre, and varies as some function of the distance from it, we have the following equation to determine the orbit,

$$h^2\left(\frac{d^2u}{d\omega^2}+u\right)\pm\frac{F}{u^2}=0. \qquad (a)$$

From this equation it is *theoretically* possible to deduce an equation between u and ω by the integral calculus. Such deduction is difficult in most cases, and frequently impossible, as the integrations required often transcend the limits of the *calculus* in its present state. There are, however, many cases in which the nature of the orbit may be determined without much difficulty, and these we propose to make the object of this part of the present Treatise.

PROP. XL.

(154.) *To determine the orbit when the force varies inversely as the square of the distance from the centre.*

In this case $F = -\mu u^2$, and consequently equation (a) becomes

$$h^2.\left(\frac{d^2u}{d\omega^2} + u\right) \pm \mu = 0. \qquad (b)$$

(155.) If the orbit described be a circle, we have $h^2u = \mu$, and consequently $h^2 = \mu r$. Since $h^2 = v^2r^2$, we have also $v^2 = \frac{\mu}{r}$, but the velocity in falling from infinity with this law of force in a right line is $\frac{2\mu}{r}$, and therefore the velocity in the circle is half that acquired in falling from infinity.

(156.) Multiplying equation (b) by $2\,du$, and integrating, we obtain

$$h^2.\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\mu u + k^2 = 0, \qquad (c)$$

in which k^2 denotes an arbitrary constant, which may be either cypher, positive, or negative. Since $h^2.\left(\frac{du^2}{d\omega^2} + u^2\right)$ is obviously the square of the velocity, and since $2\mu u$ is the velocity from infinity, we have in the first case the square of the velocity equal to that from infinity; in the second, less; and in the third, greater. Conformably to this distinction we may enumerate three distinct species of curves described with this law of force.

(157.) From equation (c) we have

$$d\omega=\frac{h.\,du}{\sqrt{2\mu u-h^2u^2-k^2}}=\frac{du}{\sqrt{\left(\frac{\mu^2}{h^4}-\frac{k^2}{h^2}\right)-\left(\frac{\mu}{h^2}-u\right)}}.$$

The integral of this gives us

$$\omega-v=\cos.^{-1}\frac{\mu-h^2u}{\sqrt{\mu^2-h^2k^2}}.$$

Hence substituting $\frac{1}{r}$ for u, and solving for r, we obtain

$$r=\frac{h^2}{\mu-\sqrt{\mu^2-h^2k^2}.\cos.(\omega-v)}. \qquad (d)$$

Assume $r=\frac{a.(1-e^2)}{1-e\cos.(\omega-v)}$, which is the equation of a conic section, the focus being the pole. This equation is of the same form as equation (d), and may be identified with it by making $a.(1-e^2)=\frac{h^2}{\mu}$, and $e=\frac{\sqrt{\mu^2-h^2k^2}}{\mu}$.

(158.) The elements sufficient to determine the orbit in *nature* and in *form* are five:—the law of force; the absolute force, or force μ at the unit of distance; the initial distance; the velocity of projection; and angle or direction of projection. Two of these elements, the law of force and absolute force, have already been used in determining the *nature* of the orbit, and the *form* we shall next proceed to determine, by introducing the other three. We shall apply these elements to determine the constants k and h in each of the species above mentioned.

(159.) *Species* 1.—Let $k^2>0$. It is obvious that the curve in this case must be an ellipse, since $e < 1$. To determine it, let v' be the velocity at the commencement of the motion, l the distance from the centre of force, and θ the angle of projection; then by equation (c), we have

$$k^2=\frac{2\mu}{l}-v'^2;$$

we have also

$$h=v'.l.\sin.\theta,$$

and consequently substituting these values for h and k^2, we have the values of a and e in terms of the initial conditions of the motion, viz.

$$a=\frac{\mu l}{2\mu-lv'^2}, \quad \text{and } e=\sqrt{1-\frac{v'^2 l^2.\sin.^2\theta}{\mu^2}.\left(\frac{2\mu}{l}-v'^2\right)},$$

or substituting for v'^2 in terms of a, μ, and l, we have

$$e=\sqrt{1-\left(\frac{2l}{a}-\frac{l^2}{a^2}\right).\sin.^2\theta}.$$

Vide Mec. Cel. p. 191.

It appears from the above results that v', l, and μ being given, a is determined, and must ever remain invariable; and (which is rather a singular circumstance) it is independant of the angle of projection. It also appears that e is determined, and must remain invariable; but it is not, like a, independant of θ.

(160.) In the value of h, substituting for v' in terms of μ, l, and a, we obtain

$$h^2=\frac{u.(2a-l).l}{a}.\sin.^2\theta,$$

but

$$\sin.^2\theta=\frac{a^2.(1-e^2)}{(2a-l).l}$$

by the properties of the ellipse, hence

$$h^2=\mu a.(1-e^2).$$

(161.) Since $\frac{du^2}{d\omega^2}=\frac{dr^2}{h^2dt^2}$, we have from equation (*c*) the following equation for determining t, putting for h^2 and k^2 their values,

$$dt=\sqrt{\frac{a}{\mu}}\cdot\frac{r\,dr}{\sqrt{a^2e^2-(r-a)^2}}\cdot$$

Precisely the same equation that has been integrated in the direct problem, Prop. 31, Art. (97).

The various affections of a body describing this curve, having been fully discussed in Prop. (31.), it is needless to repeat them here. The above determination of v', h, and t are sufficient to shew that the elements and conditions of any curvilinear motion are equally determinable, whether we commence with the *direct* or the *inverse* method of central forces.

(162.) *Species* 2.—Let $k^2=0$. In this case the curve is obviously a parabola referred to its focus, and the parameter of which is $\frac{2h^2}{\mu}\cdot$ Denoting this parameter by 4 D, we have

$$\mathrm{D}=\frac{v'^2l^2.\sin.^2\theta}{\sqrt{2\mu}},\qquad\text{and } h=\sqrt{2\mu\mathrm{D}}.$$

The velocity, time, &c. may be determined as above, or as in Prop. 31, Art. (99).

(163.) *Species* 3.—Let $k^2<0$. The curve is an hyperbola, and the values of the different elements are determined *mutatis mutandis*, as in *Species* 1, or in Prop. 31.

(164.) Since the square of the velocity from infinity is $\frac{2\mu}{r}$, it is obvious from equation (*c*) that in the ellipse the square of the velocity is less than that from infinity, and is that which would be acquired in falling through the space $(2a-r)$. In the parabola, for the same reason, the square of the velocity is that which would be acquired in falling from infinity. In the hyperbola it is greater than that from infinity, by the quantity $\frac{\mu}{a}$.

(165.) Since h is equal to $\mu.\frac{1}{2}L$, by Art. (87.), it is obviously the same as in a circle whose radius is $\frac{1}{2}L$. The mean angular velocity is equal to that in a circle whose area is equal to that of the ellipse described, and consequently the angular velocity in the ellipse becomes equal to the mean angular velocity at a distance equal to $a.\sqrt[4]{1-e^2}$. The moving point, when it arrives at the extremity of the focal ordinate, would describe a circle, but that the direction of the motion is not perpendicular to the *radius vector*; and arriving at the extremity of the axis it would describe a circle, but that the velocity, &c. are not those requisite for circular motion.

It may not be unnecessary to remark, that if μ be negative, or the force repulsive, k^2 is essentially negative in equation (c), and consequently the curve described an hyperbola. The curve is the opposite hyperbola to that described with an attractive force.

PROP. XLI.

(166.) *To determine the orbit when the force varies directly as the distance from the centre.*

Writing $\frac{\mu}{u}$ in formula (26), Sect. III. we have

$$h^2\left(\frac{d^2u}{d\omega^2}+u\right)\pm\frac{\mu}{u^3}=0. \qquad (a)$$

Using the upper sign when the force is attractive, and the lower when it is repulsive. Multiplying by $2du$, and integrating, we have

$$h^2\left(\frac{du^2}{d\omega^2}+u^2\right)\pm\frac{\mu}{u^2}-k^2=0. \tag{b}$$

Hence

$$d\omega=\frac{udu}{\sqrt{k^2u^2-u^4\mp\frac{\mu}{h^2}}}=\frac{udu}{\sqrt{\left(\frac{1}{4}k^4\mp\frac{\mu}{h^2}\right)-\left(\frac{1}{2}h^2-u^2\right)^2}}.$$

Integrating this, we obtain

$$\omega-v=\tfrac{1}{2}.\cos.^{-1}\frac{\frac{1}{2}k^2-u^2}{\sqrt{\frac{1}{4}k^4\mp\frac{\mu}{h^2}}}.$$

Or taking the inverse fun[illegible]s, we find

$$\sqrt{\tfrac{1}{4}k^4\mp\frac{\mu}{h^2}}\cdot\cos.2.(\omega-v)=\tfrac{1}{2}.k^2-u^2.$$

Hence introducing r for $\frac{1}{u}$ we find, after reduction,

$$r^2=\frac{2h}{k^2h+\sqrt{k^4h^2\mp4\mu}-2\sqrt{k^4h^2\mp4\mu}.\cos.^2(\omega-v)}. \tag{c}$$

The polar equation of a conic section, the centre being the pole, is

$$r^2=\frac{a^2(1-e^2)}{1-e^2\cos.^2(\omega-v)}.$$

Identifying this with equation (*c*), we can determine the semi-axis a, and eccentricity e, by the equations $a^2(1-e^2)=\frac{2h}{k^2h+\sqrt{k^4h^2\mp4\mu}}$, and $e^2=\frac{2\sqrt{k^4h^2\mp4\mu}}{k^2h+\sqrt{k^4h^2\mp4\mu}}$.

The orbit described shall be an ellipse or hyperbola, according as $e^2<$ or >1, that is, according as we use the upper or the lower sign to the quantity μ. This is the same as to say, if the force be *attractive* the orbit is an ellipse, and if *repulsive* an hyperbola.

If k^2 be negative in equation (b), μ must be negative, and consequently positive in the value of e^2, and therefore, the curve an hyperbola. This points out a remarkable distinction between this Prop. and the preceding; since the difference of species in the preceding depends upon the sign of k^2 without any regard to μ, and the reverse is the case in the present problem.

We shall now proceed to make use of the conditions to determine the constants k and h in each of the above species.

(167.) *Species* 1.—Let $\mu > 0$ in *eq* (b). The curve is in this case a central ellipse, since $e < 1$. Let v', l, and θ denote the same as in the [illegible]ceding Prop., then since $k^2.\left(\frac{du^2}{d\omega^2}+u^2\right)$ is the square of the velocity for any value of u, we have

$$k^2 = \pm\mu l^2 + v'^2,$$

and we have also

$$h = v'.l.\sin.\theta.$$

Introducing these values for h and k^2 in the above expressions for $a^2(1-e^2)$ and e^2, we could determine the axes. This may be effected more simply as follows:— since at the extremities of the axes $dr=0$, we have in equation (b), $\frac{du^2}{d\omega^2}=0$, and consequently the equation becomes

$$r^4 - \frac{k^2}{\mu}\cdot r^2 + \frac{h^2}{\mu} = 0,$$

or introducing for k^2 and h their values, we have the following equation to determine the axes,

$$r^4 - \frac{v'^2+\mu l^2}{\mu}\cdot r^2 + \frac{v'^2 l^2.\sin.^2\theta}{\mu} = 0.$$

(168.) To determine the position of the axes, we need but substitute from this equation for r^2 in equation (c), whence we know $(\omega - v)$, but we know ω for the commencement of the motion, and therefore we can find v.

(169.) We can now with the greatest facility proceed to determine the various circumstances of the motion in terms of the semi-axes. Let them be denoted by a and b as usual, then by the equation $r^4 - \frac{k^2}{\mu} \cdot r^2 + \frac{h^2}{\mu} = 0$, we have

$$a^2 + b^2 = \frac{k^2}{\mu},$$

and consequently the v[illegible] at any time is had from equation (b), viz.

$$v = \sqrt{\mu.(a^2 + b^2 - r^2)} = b'.\sqrt{\mu},$$

as in Art. (108).

We have also $a^2b^2 = \frac{h^2}{\mu}$, and consequently $h = ab.\sqrt{\mu}$, as in Art. (109).

Since $\frac{du^2}{d\omega^2} = \frac{dr^2}{h^2dt^2}$, we have from equation (b), substituting for k^2 and h their values, the following equation to determine t,

$$dt = \frac{1}{\sqrt{\mu}} \cdot \frac{rdr}{\sqrt{-r^4 + (a^2 + b^2)r^2 - a^2b^2}};$$

write in this expression $b^2 + e^2x^2$ for r^2, and the result gives

$$dt = \frac{1}{\sqrt{\mu}} \cdot \frac{dx}{\sqrt{a^2 - x^2}},$$

the same as has been integrated in Art. (110.), and obtained by a different process.

Species 2, admits of a precisely similar discussion, recollecting that the curve described in this case is an hyper-

From this equation it is *theoretically* possible to deduce an equation between u and ω by the integral calculus. Such deduction is difficult in most cases, and frequently impossible, as the integrations required often transcend the limits of the *calculus* in its present state. There are, however, many cases in which the nature of the orbit may be determined without much difficulty, and these we propose to make the object of this part of the present Treatise.

PROP. XL.

(154.) *To determine the orbit when the force varies inversely as the square of the distance from the centre.*

In this case $F = -\mu u^2$, and consequently equation (a) becomes

$$h^2.\left(\frac{d^2u}{d\omega^2} + u\right) \pm \mu = 0. \qquad (b)$$

(155.) If the orbit described be a circle, we have $h^2u = \mu$, and consequently $h^2 = \mu r$. Since $h^2 = v^2r^2$, we have also $v^2 = \frac{\mu}{r}$, but the velocity in falling from infinity with this law of force in a right line is $\frac{2\mu}{r}$, and therefore the velocity in the circle is half that acquired in falling from infinity.

(156.) Multiplying equation (b) by $2\,du$, and integrating, we obtain

$$h^2.\left(\frac{du^2}{d\omega^2} + u^2\right) - 2\mu u + k^2 = 0, \qquad (c)$$

in which k^2 denotes an arbitrary constant, which may be either cypher, positive, or negative. Since $h^2.\left(\frac{du^2}{d\omega^2} + u^2\right)$ is obviously the square of the velocity, and since $2\mu u$ is the velocity from infinity, we have in the first case the square of the velocity equal to that from infinity; in the second, less; and in the third, greater. Conformably to this distinction we may enumerate three distinct species of curves described with this law of force.

(157.) From equation (c) we have

$$d\omega = \frac{h.\,du}{\sqrt{2\mu u - h^2u^2 - k^2}} = \frac{du}{\sqrt{\left(\frac{\mu^2}{h^4} - \frac{k^2}{h^2}\right) - \left(\frac{\mu}{h^2} - u\right)^2}}.$$

The integral of this gives us

$$\omega - v = \cos.^{-1}\frac{\mu - h^2u}{\sqrt{\mu^2 - h^2k^2}}.$$

Hence substituting $\frac{1}{r}$ for u, and solving for r, we obtain

$$r = \frac{h^2}{\mu - \sqrt{\mu^2 - h^2k^2}.\cos.(\omega - v)}. \qquad (d)$$

Assume $r = \frac{a.(1-e^2)}{1-e\cos.(\omega-v)}$, which is the equation of a conic section, the focus being the pole. This equation is of the same form as equation (d), and may be identified with it by making $a.(1-e^2) = \frac{h^2}{\mu}$, and $e = \frac{\sqrt{\mu^2-h^2k^2}}{\mu}$.

(158.) The elements sufficient to determine the orbit in *nature* and in *form* are five:—the law of force; the absolute force, or force μ at the unit of distance; the initial distance; the velocity of projection; and angle or direction of projection. Two of these elements, the law of force and absolute force, have already been used in determining the *nature* of the orbit, and the *form* we shall next proceed to determine, by introducing the other three. We shall apply these elements to determine the constants k and h in each of the species above mentioned.

(159.) *Species* 1.—Let $k^2 > 0$. It is obvious that the revolutions. The whole time into the centre from any distance being called T, is expressed by the series

$$T=\frac{a^{2\omega}.\sec.\theta}{2\sqrt{\mu}}.\left(1-\frac{1}{a^{2\pi}}\right).\left\{1+\frac{1}{a^{2\pi}}+\frac{1}{a^{4\pi}}+\&c.\ ad\ inf.\right\}$$

The sum of this gives us

$$T=\frac{a^{2\omega}.\sec.\theta}{2\sqrt{\mu}}=\frac{r^{2}.\sec.\theta}{2\sqrt{\mu}}.$$

The space fallen through externally in this curve is infinite, and by formula (23), Sect. III., the value of Δ' is $\frac{r}{\sqrt{2}}$. Hence the locus of the points fallen to internally is a logarithmic spiral similar to the given one.

(175.) *Species* 2.—Let $\frac{\mu}{h^2}=1$ in equation (a). Integrating, we obtain

$$\frac{du}{d\omega}=k;$$

whence

$$u=k.\omega+k'.$$

Now when $u=0$, or $r=\infty$, let $\omega=0$, then the equation of the orbit is $r\omega=k$. This is the equation of the hyperbolic or reciprocal spiral.

(176.) In this case the value of h is equal to that in the equidistant circle. The velocity is greater than in the equidistant circle, for $v^2=k^2\left(\frac{du^2}{d\omega^2}+u^2\right)=\mu.(k^2+u^2)$, and therefore greater than would be acquired from infinity.

(177.) The paracentric force in this curve is cypher, and consequently the paracentric velocity is constant, as evidently appears from Art. (61). Hence the orbit described would be a circle if the angle of projection were right. The constant k is easily determined from the initial conditions of the motion, for we have

$$v'^2=\mu.\left(k^2+\frac{1}{l^2}\right),$$

whence

$$k^2=\frac{v'^2l^2-\mu}{\mu l^2}=\frac{\cot.^2\theta}{l^2}.$$

(178.) To determine the time, we have

$$hdt=r^2d\omega=-\frac{dr}{k},$$

whence

$$dt=-\frac{l\,dr}{\sqrt{\mu}.\cot.\theta}.$$

Integrating this, we find

$$t=\frac{l.\tan.\theta}{\sqrt{\mu}}.(r'-r),$$

between any required limits, r' and r. If taken between the limits ω' and ω, we have

$$t=\frac{1}{\sqrt{\mu}}.\left(\frac{1}{\omega}-\frac{1}{\omega'}\right).$$

If T denote the whole time to the centre, commencing with $\omega=2\pi$, we shall have

$$T=\frac{1}{2\pi\sqrt{\mu}}.\left\{\frac{1}{1.2}+\frac{1}{2.3}+\frac{1}{3.4}+\&c.\right\}=\frac{1}{2\pi\sqrt{\mu}}.$$

(179.) To construct the actual orbit, let o, Fig. (45), be the point of projection, C the centre of force, &c. Describe an arc $ov=k$, with the centre C and the distance Co, join Cv, and this line shall be the axis whence ω is to be reckoned. The curve has an asymptote, for let p be the perpendicular on the tangent, then $\frac{1}{p^2}=u^2+\frac{du^2}{d\omega^2}=u^2+k^2$. This when $u=0$, or $r=\infty$, gives us $p=\frac{1}{k}=l\cot.\theta$. Hence draw a right line parallel to C*v* at a distance $l\cot.\theta$, and it shall be an asymptote. The curve intersects the axis C*v* in an infinite number of points whose distances from C are in harmonic progression.

(180.) *Species* 3.—Let $\frac{\mu}{h^2}>1$, and k^2 of the form $(+\gamma^2.m^2)$. Then from equation (*b*), we have

$$\frac{du^2}{d\omega^2}=\gamma^2.(u^2-m^2),$$

and therefore

$$d\omega=\frac{du}{\gamma.\sqrt{u^2-m^2}}.$$

To integrate this, let us assume $\sqrt{u^2-m^2}=u-z$, and we readily obtain

$$\gamma d\omega=-\frac{dz}{z},$$

and consequently

$$\gamma.\omega=-\log.z+k'=-\log.\{u-\sqrt{u^2-m^2}\}+k',$$

when $u=m$ let $\omega=a$, then

$$\gamma a=-\log.m+k',$$

and therefore

$$\gamma.(\omega-a)=\log.\frac{m}{u-\sqrt{u^2-m^2}}=\log.\frac{u+\sqrt{u^2-m^2}}{m}.$$

Passing from logarithms to numbers we obtain for the equation of the orbit

$$u=\frac{m}{2}\left\{\epsilon^{\gamma.(\omega-a)}+\epsilon^{-\gamma.(\omega-a)}\right\}. \qquad (c)$$

(181.) To determine the *max.* value of r, let equation (c) be differentiated, and the result is

$$\frac{du}{d\omega}=\frac{m\gamma}{2}.\left\{\epsilon^{\gamma.(\omega-a)}-\epsilon^{-\gamma.(\omega-a)}\right\}. \qquad (d)$$

This result equated to cypher gives us $\omega=a$; and at this point the orbit has an apse for $\frac{d\omega}{du}=-\frac{r^2d\omega}{dr}=\infty$, but $\frac{rd\omega}{dr}$ is the tangent of the angle under the tangent and radius vector, and therefore at this point this angle is right. The value of u for this value of ω is m by *eq*, (c).

(182.) The value of h is less than in the equidistant circle, and the velocity is also less since $v^2=h^2\left(\frac{du^2}{d\omega^2}+u^2\right)=\mu u^2-(\mu-h^2).m^2$, and $\mu>h^2$.

(183.) To find the time we have

$$\frac{du^2}{d\omega^2}=\frac{dr^2}{h^2dt^2}=\gamma^2.(u^2-m^2)=\gamma^2\left(\frac{1}{r^2}-\frac{1}{a^2}\right),$$

(a being the apsidal distance or reciprocal of m). Hence

$$dt=\frac{ardr}{h\gamma.\sqrt{a^2-r^2}};$$

and since a is the greatest value of r, if we commence the time from the apse, dr must be negative; hence, integrating, we obtain

$$t=\frac{a}{h\gamma}.\sqrt{a^2-r^2}.$$

The whole time into the centre is $\frac{a^2}{h\gamma}$, or putting for $h\gamma$

its value, $\frac{\mu a.\cos.\theta}{\sqrt{a^2-l^2.\sin.^2\theta}}$, we have the whole time in terms of the elements of projection and the apsidal distance.

(184.) To represent these circumstances, and construct the orbit; let o, (Fig. 46.), be the point of projection, and C the centre of force. Then knowing the conditions of the motion at the point o by hypothesis, we can find the quantity m by Art. (182.), and consequently a. Draw therefore the line $Cv=a$, making the angle $oCv=a$, and the point v is the apse.

From equation (d) it appears that u diminishes from o to v, and after that increases indefinitely, hence r increases from o to v, and diminishes indefinitely afterwards. If ω be negative, it appears from equation (d) that u increases, and consequently r diminishes indefinitely.

Denoting an angle, P′Co, by $(-\omega')$, and another, PCo, by (ω), by the equation (c) the lines CP, CP′ will be equal, if

$$\varepsilon^{-\gamma.(\omega'+a)}+\varepsilon^{\gamma.(\omega'+a)}=\varepsilon^{\gamma.(\omega-a)}+\varepsilon^{-\gamma.(\omega-a)},$$

that is, if $\omega'+a=\omega-a$; or in other words, if the angles VCP, VCP′ be equal. Hence the two branches of the curve are perfectly symmetrical on each side of the line VCn, and they mutually intersect in an infinite series of points along this line on each side of C. The distance of any one of these intersections from C may be readily found; to find the n^{th}, write $n\pi$ for $(\omega-a)$ in equation c, and we have

$$r=2a.\frac{\varepsilon^{2\gamma n\pi}}{\varepsilon^{\gamma n\pi}+1},$$

n being any number in the series 0, 1, 2, 3, &c.

(185.) *Species* 4.—Let $\frac{\mu}{h^2}>1$, and k^2 of the form $(-\gamma^2m^2)$. Then from equation (b) we have

$$d\omega=\frac{du}{\gamma.\sqrt{u^2+m^2}}.$$

Integrating this as the formula in the preceding species, we obtain

$$\gamma\omega=-l.\{\sqrt{u^2+m^2}-u\}+k'.$$

When $u=0$ let $\omega=a$, then $\gamma a=-lm+k'$. Hence

$$\gamma(\omega-a)=l.\frac{\sqrt{u^2+m^2}+u}{m}.$$

Passing from logarithms to numbers, we finally obtain

$$u=\frac{m}{2}.\{\varepsilon^{\gamma(\omega-a)}-\varepsilon^{-\gamma(\omega-a)}\} \qquad (e)$$

(186.) In this case, as in the preceding, h is less than in the equidistant circle, but the velocity is greater, for we have

$$v^2=\mu u^2+(\mu-h^2).m^2.$$

(187.) To find the time, we have

$$\frac{du^2}{d\omega^2}=\frac{dr^2}{h^2dt^2}=\gamma^2.(u^2+m^2),$$

whence

$$dt=\frac{r\,dr}{h\gamma\sqrt{r^2+a^2}},$$

(a denoting the reciprocal of m, or the value of r, when $\varepsilon^{\gamma(\omega-a)}=1\pm\sqrt{2}$. Hence, integrating, we obtain

$$t=\frac{1}{h\gamma}.\sqrt{r^2+a^2}+\mathrm{C},$$

whence the time can be found within any assigned limits.

(188.) The orbit may be constructed as follows. Having determined the constants as in the preceding cases, let o, (Fig. 47.), be the point of projection; draw the line Co', making the angle $oCo' = a$, and the line Co' shall meet the curve at an infinite distance, for when $\omega = a$, $u = 0$, and therefore r is infinite. As ω increases u increases, because $\varepsilon^{\gamma \cdot (\omega - a)}$ increases and $\varepsilon^{-\gamma \cdot (\omega - a)}$ diminishes; hence r diminishes indefinitely, and therefore the curve after an infinite number of revolutions reaches the centre C.

(189.) To determine whether the curve has an asymptote; let p denote the perpendicular on the tangent, then

$$\frac{1}{p^2} = u^2 + \frac{du^2}{d\omega^2} = u^2 + \frac{\gamma^2}{4} \cdot (\varepsilon^{\gamma(\omega - a)} + \varepsilon^{-\gamma(\omega - a)})^2.$$

If $u = 0$, or $r = \infty$, we shall have by this formula $p = \frac{1}{\gamma}$, and consequently finite. Draw then a line Cm perpendicular to Co' and equal to $\frac{1}{\gamma}$, and from m draw ms parallel to Co', the line ms is an asymptote.

(190.) *Species* 5.—In the original differential equation, (a), let $\mu < h^2$. Then

$$\frac{d^2u}{d\omega^2} = -\left(1 - \frac{\mu}{h^2}\right) \cdot u,$$

multiplying by $2du$, and integrating, we obtain

$$\frac{du^2}{d\omega^2} = -\left(1 - \frac{\mu}{h^2}\right) u^2 + k^2.$$

Hence putting γ^2 for $\left(1 - \frac{\mu}{u^2}\right)$, we have

$$\gamma d\omega = \frac{du}{\sqrt{\frac{k^2}{\gamma^2} - u^2}}.$$

which being integrated we obtain

$$\gamma.(\omega-a)=\cos.^{-1}\frac{\gamma u}{k},$$

whence we have

$$r=\frac{\gamma}{k.\cos.\gamma(\omega-a)}.$$

The several conditions of the motion may be determined as in the preceding species.

(191.) The orbit may be constructed as follows: when $\omega=a$, $r=\frac{\gamma}{k}$, and $\frac{dr}{d\omega}=0$, therefore, at this point there is an apse. Let A, (Fig. 48.), be the point of projection, and A*so*$=a$, then at *o* there is an apse. When $\gamma.(\omega-a)=\frac{\pi}{2}$, r is ∞; hence drawing the line *sn*, so that the angle A*sn* be equal to $\left(\omega+\frac{\pi}{2\gamma}\right)$, it shall meet the curve at an infinite distance. The curve has an asymptote parallel to the line *sn*, for the length of the perpendicular on the tangent where $\gamma.(\omega-a)=\frac{\pi}{2}$, is k. Hence draw from *s* a line, *s*B, perpendicular to *sn* and equal to k, and draw B*v* parallel to *sn*, the line B*v* will be an asymptote.

PROP. XLIII.

(192.) *The force being inversely as the fifth power of the distance, to investigate the orbit.*

The general differential equation becomes in this case

$$\frac{d^2u}{d\omega^2}+u-\frac{\mu u^3}{h^2}=0.$$

Multiplying by $2du$ and integrating, we have

$$\frac{du^2}{d\omega^2}+u^2-\frac{\mu u^4}{2h^2}=k^2. \qquad (a)$$

This equation we shall integrate on separate hypotheses, as in the last proposition.

(193.) If the orbit described be a central circle, we shall have $h=\frac{\sqrt{\mu}}{r}$, and consequently the velocity equal to $\frac{\sqrt{\mu}}{r^2}$. This velocity is greater than that from infinity, in the ratio of $\sqrt{2}$ to 1.

(194.) *Species* 1.—Let $k^2=0$, and $\frac{\mu}{h^2} > 0$; we have then from equation (*a*)

$$d\omega=\frac{du}{u\sqrt{\frac{\mu u^2}{2h^2}-1}}.$$

In order to integrate this, let $u=h.\sqrt{\frac{2}{\mu}}.\sec.\theta$, then

$$du=\sqrt{\frac{2}{\mu}}.\frac{d\theta.\sin.\theta}{\cos.^2\theta}, \quad \text{and} \quad \sqrt{\frac{\mu u^2}{2h^2}-1}=\tan.\theta.$$

Combining these values we obtain

$$d\omega=d\theta,$$

and integrating,

$$\omega-a=\theta,$$

(a being the value of ω when $\theta=0$, or when $u.\sqrt{\frac{\mu}{2h^2}}=1$).
The equation of the curve described is consequently

$$r=\sqrt{\frac{\mu}{2h^2}}.\cos.(\omega-a).$$

This is the equation of a circle with the centre of force on the circumference, and the diameter of which makes an angle, a, with the line from the centre of force to the point of projection. The length of the diameter is $\sqrt{\frac{\mu}{2h^2}}$.

(195.) Denoting the radius of this circle by ρ, we have

$$\mu = 8\rho^2 h^2,$$

but in a central circle with the same radius ρ, we have, by Art. (58.),

$$\mu = \rho^2 h^2.$$

Consequently, if h be the same in both, that is, if they be described in the same time, the absolute forces are as 8 to 1.

(196.) The velocity can be had as usual, its value is $v = \frac{\sqrt{\mu}}{r^2 . \sqrt{2}}$, and is to the velocity in the equidistant circle as 1 to $\sqrt{2}$.

To find the time, since $\frac{du^2}{d\omega^2} = \frac{dr^2}{h^2 dt^2}$, we have

$$dt = \frac{\sqrt{\mu}}{2\sqrt{2}.\rho} \cdot \frac{r^2 dr}{\sqrt{4\rho^2 - r^2}}.$$

The integral of this gives us

$$t = \frac{\sqrt{\mu}}{2\sqrt{2}.\rho} \cdot \left\{ -\frac{r.\sqrt{4\rho^2 - r^2}}{2} - 2\rho^2 . \cos.^{-1} \frac{r}{2\rho} \right\} + \text{C}.$$

Taking this between the limits $r = 2\rho$ and $r = 0$, we get for half the periodic time

$$\frac{\pi\rho.\sqrt{\mu}}{2\sqrt{2}}.$$

The other conditions of the motion may be determined as before.

(197.) By equation (a) we have

$$\frac{du^2}{d\omega^2} = k^2 + \frac{\mu u^4}{2h^2} - u^2.$$

If the second member of this equation be a perfect square, we can readily perform the required integration. This condition gives us

$$\frac{2k^2\mu}{h^2}=1.$$

whence we have

$$\frac{du}{d\omega}=\pm\frac{h^2-u^2\mu}{h.\sqrt{2\mu}}.$$

(198.) *Species* 2.—In the preceding article let us use the sign +, and we have

$$d\omega=\frac{h.\sqrt{2\mu}.du}{h^2-u^2\mu}.$$

Integrating this as a fractional function, we readily obtain

$$\sqrt{2}.(\omega-a)=\log.\frac{h+u.\sqrt{\mu}}{h-u.\sqrt{\mu}},$$

a being what ω becomes when the second member vanishes, that is, when $u=0$.

When $\omega=a$, r is infinite; and as ω increases u must increase, and consequently r must diminish; and when ω is infinite we have $r=\frac{\sqrt{\mu}}{h}$. This is the least possible value of r, and a value that it can have only when the curve has made an infinite number of revolutions. If a circle be described with this radius, the curve shall perpetually approach but never reach it. This circle is called an asymptotic circle.

This curve has also an asymptote, for $\frac{1}{p^2}=u^2+\frac{du^2}{d\omega^2}=\frac{h^2}{2\mu}+\frac{\mu u^4}{2h^2}$. Hence, when $u=0$, or $r=\infty$, $p=\sqrt{\frac{2\mu}{h^2}}$.

(199.) *Species* 3.—Using the negative sign in Art. (197.), we have to integrate $d\omega = \frac{h.\sqrt{2\mu}.du}{\mu u^2 - h^2}$, which by the same process of integration gives us $\sqrt{2}.(\omega - a) = \log.\frac{u\sqrt{\mu} - h}{u\sqrt{\mu} + h}$, a being the value of ω when the second member vanishes, that is, when u is infinite.

(200.) When $\omega = a$, u is infinite, and consequently $r = 0$; as ω increases, u diminishes, and consequently r increases; and when ω is infinite we have $r = \frac{\sqrt{\mu}}{h}$. This circumstance indicates an asymptotic circle, within which the orbit described is altogether situated.

(201.) The construction of the orbits in these two species can be readily effected by introducing the usual conditions of projection, as in the preceding propositions.

(202.) In both these curves we have $v^2 = \frac{h^4}{2\mu} + \frac{\mu u^4}{2}$, which when $u^2 = \frac{h^2}{\mu}$ becomes μu^4, the same as in the equidistant circle by Art. (193). This is because the motion then approximates to circular motion, namely, in the asymptotic circle. In the first of these species $u < \frac{h}{\sqrt{\mu}}$ always, and consequently the velocity, is greater than in the equidistant circle. The reverse is the case in the second species. There are several mistakes in the discussions of these curves in WHEWELL's *Dynamics*, *vide* p. 52.

The first notice of the asymptotic circle in curves is due to MC. LAURIN, *vide Flux.*, *vol.* 2., p. 259. His discovery, however, was restricted to the case in which the force varies inversely as the fifth power of the distance. SIMPSON was the first who gave a general method of ascertaining the conditions requisite that any orbit may have such a circle.—*Matth. Diss.*, p. 164, &c.

The following method of determining these conditions is considerably different from any that has been hitherto given.

PROP. XLIV.

(203.) *To determine generally the conditions requisite, that an orbit may have an asymptotic circle.*

The force being supposed to be inversely as the n^{th} power of the distance, let r' be any known value of the radius vector, and A the velocity at the distance r'. Let the radius of the asymptotic circle be r, then the velocity at the distance r shall be had by the formula, Art. (38.),

$$v^2=A^2+\frac{2\mu}{n-1}\cdot\left\{\frac{1}{r^{n-1}}-\frac{1}{r'^{n-1}}\right\}.$$

Let A be e times the velocity in a circle at the distance r', then

$$A^2=\frac{\mu e^2}{r^{n-1}},$$

therefore,

$$v^2=\frac{\mu e^2}{r'^{n-1}}+\frac{2\mu}{n-1}\cdot\left\{\frac{1}{r^{n-1}}-\frac{1}{r'^{n-1}}\right\}.$$

Since v must be equal to the velocity in a circle, we have

$$v^2=\frac{\mu}{r^{n-1}}.$$

Hence

$$\frac{1}{r^{n-1}}=\frac{e^2}{r'^{n-1}}+\frac{2}{(n-1).r^{n-1}}-\frac{2}{(n-1).r'^{n-1}}.$$

Hence

$$\frac{e^2.(n-1)-2}{(n-1).r'^{n-1}}=\frac{(n-3)}{(n-1).r^{n-1}}.$$

Therefore,

$$r^{n-1}=\frac{(n-3).(r'^{n-1})}{e^2.(n-1)-2},$$

and

$$r=r'.\sqrt[n-1]{\frac{n-3}{e^2.(n-1)-2}}.$$

At this distance the velocity is precisely that requisite, in order that a circle may be described with a radius r; but it is moreover necessary that the angle of projection should be right. This condition will enable us to determine the relation between the velocity and direction of projection at the distance r'. We have obviously

$$h^2=r'^2.\sin.^2\theta.\,A^2=\frac{\mu e^2.\sin.^2\theta}{r'^{n-3}},$$

and since the angle of projection is right at the distance r, we have

$$h^2=\frac{\mu}{r^{n-3}}.$$

Equating these values of h^2, we find

$$r^{n-3}=\frac{r'^{n-3}}{e^2.\sin.^2\theta},$$

therefore,

$$r=r'.\sqrt[n-3]{\frac{1}{e^2.\sin.^2\theta}}=r'.\sqrt[n-1]{\frac{n-3}{e^2.(n-1)-2}}.$$

Hence solving, we find

$$\sin.^2\theta=\frac{1}{e^2}.\left\{\frac{(n-1)e^2-2}{n-3}\right\}^{\frac{n-3}{n-1}}.$$

The circle will be exterior to the orbit described if

$e^2(n-1)-2 < n-3$, or if $e^2 < 1$. The contrary is the case if $e^2 > 1$.

PROP. XLV.

(204.) *To determine for what laws of force an asymptotic circle can exist.*

This is only to determine in what case the equation $\sin.^2\theta = \frac{1}{e^2}.\left(\frac{(n-1)\,e^2-2}{n-3}\right)^{\frac{n-3}{n-1}}$ is possible. Differentiating this, considering e variable, we have

$$d.\sin.^2\theta = \frac{4\,de}{e^3}.\frac{1-e^2}{n-3}.\left(\frac{(n-1)\,e^2-2}{n-3}\right)^{\frac{-2}{n-1}};$$

this differential will be positive or negative according as $\frac{1-e^2}{n-3}$ is positive or negative, because $\left(\frac{(n-1)\,e^2-2}{n-3}\right)^{\frac{-2}{n-1}}$ must be positive, otherwise $\sin.^2\theta$ cannot be so. But when $e=1$, $\frac{1-e^2}{n-3}$ will be cypher, and $\frac{1}{e^2}.\left(\frac{(n-1)\,e^2-2}{n-3}\right)^{\frac{n-3}{n-1}} = 1$, which is obviously a *maximum* or *minimum* value of this function, a *max.* when $n-3$ is positive, because then the differential as long as $e < 1$ will be positive, and afterwards negative, and a *min.* when $n-3$ is negative, since the same differential will be first negative and then positive. Now unity can only be a *max.* value of $\sin.^2\theta$, and consequently, the former case alone can hold; hence, $n-3$ must be positive, or in other words, the force must vary inversely as some power of the distance higher than the third.

PROP. XLVI.

(205.) *To investigate the orbit when the force is inversely as the n^{th} power of the distance.*

In this case we have

$$\frac{d^2u}{d\omega^2}+u-\frac{\mu u^{n-2}}{h^2}=0.$$

Or, multiplying by $2du$, integrating, and solving for $d\omega$, we find

$$d\omega=\frac{du}{\sqrt{k^2-u^2+\frac{2\mu u^{n-1}}{(n-1)h^2}}}. \qquad (a)$$

(206.) The constant k^2 may be found by knowing the velocity of projection; for at the commencement of the motion, or when $u=\frac{1}{l}$, let the velocity be v': then since, in general, $v^2=h^2.\left(u^2+\frac{du^2}{d\omega^2}\right)=\frac{2\mu.u^{n-1}}{n-1}+h^2k^2$, we can find k^2 by substituting v' for v, and $\frac{1}{l}$ for u.

(207.) The square of the velocity acquired by falling from infinity with this law of force is $\frac{2\mu.u^{n-1}}{n-1}$, let v'^2 be ε^2 times this, then we shall have $k^2=\frac{\varepsilon^2-1}{h^2}.\frac{2\mu}{(n-1)l^{n-1}}$; and introducing this value in equation (a), this equation becomes, writing u' for $\frac{1}{l}$,

$$d\omega=\frac{du}{\sqrt{\frac{(\varepsilon^2-1)}{h^2}.\frac{2\mu}{n-1}.u'^{n-1}-u^2+\frac{2\mu.u^{n-1}}{(n-1)h^2}}}. \qquad (b)$$

(208.) This equation may be represented under a different form, as follows:—the velocity in a circle at the distance l is $\sqrt{\frac{\mu}{l^{n-1}}}$, let $v' = q.\sqrt{\frac{\mu}{l^{n-1}}}$, then to determine the constant k^2, we have

$$\frac{q^2\mu}{l^{n-1}} = \frac{2\mu}{(n-1).l^{n-1}} + h^2k,$$

and consequently the equation (a) becomes,

$$d\omega = \frac{h\,du}{\sqrt{\mu u'^{n-1}\left(q^2 - \frac{2}{n-1}\right) - h^2u^2 + \frac{2\mu}{n-1}\cdot u^{n-1}}}. \qquad (c)$$

(209.) At an apse we have $\frac{du}{d\omega} = 0$, and consequently from equation (b)

$$\frac{(\varepsilon^2-1)}{h^2}\cdot\frac{2\mu u'^{n-1}}{n-1} - u^2 + \frac{2\mu.u^{n-1}}{(n-1)h^2} = 0.$$

The values of u deduced from this give us the distances of the apsides from the centre of force. There may be four such values when n is even, two positive and two negative; but only two of them can hold, for the other two are introduced by squaring. If n be odd, there are three possible roots; but as some must have been introduced by squaring, there can be no more than two apsidal distances in this case also. There may be, however, an infinite number of apsides.

The equation in this article is totally insufficient for the determination of the apsides in any case, as it only determines the apsidal distances, but not the angle between. The angle between the apsides may be investigated either *directly* or *indirectly*: directly, by investigating the form of the orbit described; and indirectly, by a very elegant method first given by Dr. Brinkley in the 'Irish Transactions for 1801.' We shall first give some examples of the former method.

PROP. XLVII.

(210.) *Let the force vary inversely as the square, and inversely as the cube of the distance, to investigate the orbit and determine the apsides.*

Introducing for F in the formula (26), Art. (75.), $\mu u^2+\mu' u^3$, we find

$$\frac{d^2u}{d\omega^2}+u-\frac{\mu}{h^2}-\frac{\mu'.u}{h^2}=0.$$

Multiplying this by $2du$, integrating and solving for $d\omega$, we find, using γ^2 for $1-\frac{\mu'}{h^2}$,

$$d\omega=\frac{h\,du}{\sqrt{2\mu u-h^2\gamma^2u^2-k^2}}.$$

From this, integrating as in Art. (157.), we readily obtain

$$r=\frac{h^2\gamma^2}{\mu-\sqrt{\mu^2-h^2\gamma^2k^2}.\cos.\gamma(\omega-v)}.$$

(211.) This orbit may be constructed as follows. Let a focal ellipse be described, such that $a(1-e^2)=\frac{h^2\gamma^2}{\mu}$, and $e=\frac{\sqrt{\mu^2-h^2\gamma^2k^2}}{\mu}$, then drawing any radius vector of this ellipse making an angle, θ, with the axis major, if we draw a line equal to this radius vector, and making an angle $\frac{\theta}{\gamma}$, the extremity of this line will be a point in the locus.

If $\gamma < 1$, the curve thus described will fall without the ellipse, and if $\gamma > 1$, within it.

(212.) There is an apse in this orbit answering to the same apsidal distance as in the ellipse, but the angles between the apsides is not the same. The first apse will occur after revolving through an angle $\frac{\pi}{\gamma}$; the second after an angle $\frac{2\pi}{\gamma}$; and the n^{th} after revolving through an angle $\frac{n\pi}{\gamma}$.

If $\gamma > 1$, or in other words, if the part of the force μu^3 be subtractive, the body will arrive at an apse before it has described 180°, or before it would have arrived at an apse if it described an ellipse; if $\gamma < 1$, the reverse holds. In the former case the apsides of the elliptic orbit may be said to be *regressive*, in the latter, *progressive*.

(213.) In the case where $\mu' u^3$ is additive, since $\gamma = \sqrt{1 - \frac{\mu'}{h^2}}$, if $\mu' = h^2$, the apse is not arrived at until after an infinite number of revolutions, and the equation of the orbit is

$$r = \frac{2h^2}{\mu . (\omega - v)^2 + k},$$

the value of r when $\omega = v$ being $\frac{2h^2}{k}$.

(214.) If in this case $k^2 < \mu'$, the equation to be integrated is

$$d\omega = \frac{h\,du}{\sqrt{\gamma^2 u^2 + 2\mu u - k^2}};$$

from the integration of this it might be shewn that a body will fall into the centre without arriving at a second apse.

PROP. XLVIII.

(215.) *The force being represented by any function of the distance, to find an approximate value for the angle between the apsides when the orbit is nearly circular.*

If a body be projected at right angles to the radius vector with a velocity nearly that in a circle at the same distance and with the same force, the orbit described will be nearly a circle. The angle between the apsides depends, in general, upon the velocity and law of force. However, as the velocity approaches to that in a circle at the same distance, the angle between the apsides tends more nearly to a certain limit. This limit it can never reach, because when the velocity is equal to that in the equidistant circle, the aspidal distances are equal, a circle is the curve described, and every point is in fact an apse. To find this limiting angle is the object of this proposition, as it will serve to indicate the angle between the apsides when the difference of the apsidal distances is small. Let $F = u^2\phi(u)$, then the equation Art. (15.) gives us

$$\frac{d^2u}{d\omega^2} + u - \frac{\phi(u)}{h^2} = 0.$$

At the point when the direction of projection is at right angles to the radius vector, let $u = c$, and for any other point let $u = c + z$, z being a small quantity. Then by Taylor's theorem

$$\phi u = \phi c + \frac{d.\phi c}{dc} \cdot z + \frac{d^2\phi c}{dc^2} \cdot \frac{z^2}{1.2} + \&c.$$

Then v and v' being the velocities in the curve and equidistant circle, let $v^2 : v'^2 :: 1 : 1+\delta$; and since $v'^2 = F.\, r = c.\, \phi(c)$, we have

$$v^2 = \frac{c.\phi(c)}{1+\delta},$$

and consequently

$$h^2 = \frac{v^2}{c^2} = \frac{\phi(c)}{c.(1+\delta)}.$$

Substituting in the original equation we have

$$\left.\begin{array}{l}\dfrac{d^2z}{d\omega^2}+c+z-\dfrac{(1+\delta).c}{\phi(c)}\cdot\left\{\phi(c)+\dfrac{d.\phi(c)}{dc}\cdot\dfrac{z}{1}+\right.\\ \qquad\left.\dfrac{d^2\phi(c)}{dc^2}\cdot\dfrac{z^2}{1.2}+\&c.\right\}\end{array}\right\}=0.$$

or,

$$\left.\begin{array}{r}\dfrac{d^2z}{d\omega^2}+\left(1-\dfrac{c.d\phi(c)}{\phi(c).dc}\right)z-\dfrac{c.d^2\phi(c)}{\phi(c)}\cdot\dfrac{z^2}{1.2}-\&c.\\ -c\delta-\dfrac{cd\phi(c)zd}{\phi(c).dc}-\&c.\end{array}\right\}=0.$$

And when the orbit becomes indefinitely nearly a circle, δ and z both tend to evanescence; hence z^2, $z\delta$, &c. may be omitted in comparison with z. Hence

$$\frac{d^2z}{d\omega^2}+\left(1-\frac{c.d\phi(c)}{\phi(c).dc}\right).z-c\delta=0.$$

Making $1-\frac{c.d.\phi(c)}{\phi c.dc}=\gamma^2$, this equation may be integrated as in the last proposition, and the angle between the apsides $=\dfrac{\pi}{\sqrt{1-\dfrac{c.d\phi(c)}{\phi(c).dc}}}$.

(216.) Let the force be as any power of the distance, then $P=\mu.u^n=u^2.\mu.u^{n-2}$, hence, $\phi(u)=\mu.u^{n-2}$, and $\frac{d.\phi(u)}{du}=\mu.(n-2).u^{n-3}$. Hence we have

$$\gamma=\sqrt{3-n},$$

and consequently the angle between the apsides $=\frac{\pi}{\sqrt{3-n}}$.

(217.) Hence it appears that when $n=3$, the orbit, if nearly circular, never can have two apsides. Prop. (42) shews that in no case can it have two. If $n=-1$, $A=\frac{\pi}{2}$, which agrees with Prop. (41). When $n=1$, $A=\frac{\pi}{\sqrt{3}}$. When $n=2$, $A=\pi$, as in Prop. (40). When $n>3$ the expression is impossible, and in fact in such cases if the body leave one apse it never can reach another.

(218.) Let the force be as any two powers of the distance, then $F=\mu.u^n+\mu'.u^{n'}=u^2(\mu u^{n-2}+\mu'.u^{n'-2})$, therefore,

$$\phi(u)=\mu u^{n-2}+\mu'.u^{n'-2},$$

and consequently

$$\gamma^2=\frac{(3-n)\mu c^{n-2}+(3-n')\mu' c^{n'-2}}{\mu c^{n-2}+\mu'.c^{n'-2}},$$

and therefore A is known.

(219.) If $n=2$, and $n'=-1$, we have $\gamma^2=\frac{\mu+4\mu'.c^{-3}}{\mu+\mu'.c^{-3}}$, and therefore,

$$A=\pi\sqrt{\frac{\mu+\mu'c^{-3}}{\mu+4\mu'.c^{-3}}}.$$

If the force $\frac{\mu}{u}$ be subtractive, this expression becomes

$$A \underset{g}{=} \pi \sqrt{\frac{\mu-\mu' c^{-3}}{\mu-4\mu' c^{-3}}}.$$

The method here given exactly agrees with that of Newton in his ninth section, and gives the limit to the same degree of accuracy. But the method of Dr. Brinkley, which we shall now proceed to set before the reader, is much superior, in that the orbit is thereby constructed, and the angles between the apsides determined to any degree of accuracy.

(220.) This method consists in determining the orbit for any law of force, when we previously know that there are two apsides, and in this case the orbit may be always determined by a series of sines of multiple arcs, converging by the powers of the eccentricity. From hence the angle between the apsides is immediately determined, and thus we obtain not only all that is determined in the last Prop. of the ninth section of the *principia*, but also the motion of the apsides for eccentric orbits. The method in the ninth section gives only the limit of the motion of the apsides, and cannot be applied to find this motion in eccentric orbits. The limit is, in fact, never required, for then the orbit is a circle, but the motion before it has arrived at this limit. The motion, indeed, approximates indefinitely to the limit, but this is not so evident from the method of Newton, as from it we *only* know the limit, and nothing of its antecedent state.

It has been shewn already that when the force varies as any function of the distance, an orbit can have only two different distances of the apsides; this also readily appears from the fact, that the orbit must be similar on each side of an apse. The law of force also readily shews whether any orbit described by that law can have two apsides by comparing it with the law of the centrifugal force, which always varies in the inverse triplicate ratio of the distance. Therefore, if the force vary in any direct ratio, or in any inverse ratio less than the triplicate, the orbits described may have two apsides.

PROP. XLIX.

(221.) *The centripetal force varying inversely as the square of the distance, to determine the orbit when it has two apsides.*

Let the distances of the apsides be $a+c$ and $a-c$ respectively, and let the height from which the body should fall to acquire the velocity at the distance $(a+c)$ be z, then by Props. (1 and 40.) of the Principia, we have

$$(a+c)^2 : (a-c)^2 :: \frac{1}{a-c}-\frac{1}{z} : \frac{1}{a+c}-\frac{1}{z},$$

whence

$$z=2a.$$

Let the radius vector at any instant be r, ω the angular distance from perihelion, and p the perpendicular on the tangent, then

$$p^2 : (a+c)^2 :: \frac{1}{a+c}-\frac{1}{2a} : \frac{1}{r}-\frac{1}{2a},$$

whence

$$p^2=\frac{r}{2a-r}(a^2-c^2).$$

Since $\sin.\theta=\frac{p}{r}=\frac{r\,d\omega}{\sqrt{dr^2+r^2d\omega^2}}$, we have

$$d\omega=\frac{p\,dr}{r.\sqrt{r^2-p^2}}=\frac{dr}{r.\sqrt{\frac{r^2}{p^2}-1}},$$

or substituting for p^2 its value, we have

$$d\omega = \frac{dr}{r.\sqrt{\frac{r.(2a-r)}{a^2-c^2}-1}}.$$

This is obviously the differential of the equation

$$r = \frac{a(1-e^2)}{1+e\cos.\omega},$$

which is the equation of a focal ellipse, ω being the angle from perihelion or the mean anomaly.

For the subsequent proposition it is necessary to solve the following Lemma.

LEMMA.

(222.) *The aphelion distance of an ellipse being* $1+e$, *and the perihelion* $1-e$, *and any other distance* $1+y$, *it is required to express the value of* y *in a series of cosines of multiples of the true anomaly counted from the aphelion, and powers of the eccentricity.*

By the equation of the ellipse,

$$+y = \frac{1-e^2}{1-e.\cos.\omega},$$

whence

$$y = e.\frac{\cos.\omega - e}{1-e.\cos.\omega}.$$

Now $\frac{1}{1-e\cos.\omega} = 1 + e\cos.\omega + e^2.\cos.^2\omega + e^3.\cos.^3\omega + \&c.$

whence

$$y = -e^2 + (e-e^3)\cos.\omega + e^2.\cos.^2\omega + e^3.\cos.^3\omega + \&c.$$

$$= -\frac{e^2}{2} + \frac{e-e^3}{4}\cdot\cos.\omega + \tfrac{1}{2}e^2.\cos.2\omega + \tfrac{1}{4}e^3.\cos.^3\omega + \&c.$$

This series might be continued *ad libitum*, by observing the relation of the terms to each other, but the above is sufficient for the present. The powers of y may be had from the above,

$$y^2 = \frac{e^2}{2} - \frac{e^3}{2} \cdot \cos. \omega + \frac{e^2}{2} \cdot \cos. 2\omega + \frac{e^3}{2} \cdot \cos. 3\omega,$$

$$y^3 = \frac{3}{8} \cdot e^3. \cos. \omega + \frac{1}{4} \cdot e^3. \cos. 3\omega, \text{ \&c.}$$

&c. &c.

PROP. L.

(223). *The centripetal force varying partly in the inverse duplicate, and partly in the direct simple ratio of the distance, to determine the orbit when that orbit has two apsides.*

Let the greatest distance, SA, (Fig. 49.) be denoted by $1+e$; and the least, SB, by $1-e$; and the angle, ASP, by θ. Let the force at P be expressed by $\frac{1}{r^2} - cr$, and let z denote the height from which a body should fall to acquire the velocity at A.

The square of the velocity at P is readily found, by Arts. (35 and 39.), to be $\frac{1}{r} - \frac{1}{z} - \frac{1}{2} c (z^2 - r^2)$: and since the velocities at A and B are inversely as the distances, we have

$$\frac{1}{1+e} - \frac{1}{z} - \tfrac{1}{2} c.(z^2 - (1+e)^2) : \frac{1}{1-e} - \frac{1}{z} - \tfrac{1}{2} c.(z^2 - (1-e)^2) :: (1-e)^2 : (1+e)^2,$$

whence,

$$\frac{1}{z} - \tfrac{1}{2} c z^2 = \tfrac{1}{2} - c.(1+e^2),$$

and consequently comparing the squares of the velocities at the points A and P, we have

$$\frac{1}{r}-\tfrac{1}{2}-\tfrac{1}{2}c(2+2e^2-r^2) : \frac{1}{1+e}-\tfrac{1}{2}-\tfrac{1}{2}c(2+2e^2-(1+e)^2) :: (1+e)^2 : p^2,$$

whence

$$\frac{r^2}{p^2}=\frac{2r-r^2-cr^2(2+2e^2-r^2)}{1-e^2-c(1-e^2)^2}.$$

Hence,

$$d\theta=\frac{-dr}{r.\sqrt{\dfrac{r^2}{p^2}-1}}=\frac{-dr}{r.\sqrt{\dfrac{2r-r^2-cr^2(2+2e^2-r^2)}{1-e^2-c(1-e^2)^2}-1}}.$$

The integral of this expression may be approximated to by help of an ellipse, in which the greatest and least focal distances are $1+e$ and $1-e$, and ω the angle between the greatest distance $1+e$, and any other distance r. In this ellipse, as in the preceding Prop., we have

$$d\omega=\frac{-dr}{r.\sqrt{\dfrac{2r-r^2}{1-e^2}-1}}.$$

whence we have

$$d\theta=d\omega.\sqrt{\frac{1-c.(1-e^2)}{1-c.\dfrac{2r^2+2e^2r^2-r^4-(1-e^2)^2}{2r-r^2-1+e^2}}}.$$

Let $\frac{P}{Q}$ denote the fraction which occurs in the denominator of this expression, then since two of the roots of the equation $P=0$, viz. $1+e$ and $1-e$, are also roots of the equation $Q=0$, it follows that Q is a divisor of P.

Accordingly we find $\frac{P}{Q}=(r+1)^2-e^2=(2+y)^2-e^2$, putting $1+y=r$, hence

$$d\theta=d\omega.\sqrt{\frac{1-c}{1-4c+ce^2}}.\frac{1}{\sqrt{1-c.\frac{2y+y^2}{1-4c+ce^2}}}.$$

Let $1-4c+ce^2=B$, then by expanding the second factor,

$$d\theta=d\omega.\sqrt{\frac{1-c}{B}}.\left\{1+\frac{c}{2B}.(2y+y^2)+\frac{3c^2.y^2}{2B^2}+\&c.\right\},$$

regarding only the second power of the eccentricity. Substituting in this their values for y, y^2, &c. from the Lemma, we have, by integrating

$$\theta=\omega.\sqrt{\frac{1-c}{B}}.\left\{1-\frac{\frac{1}{4}ce^2-ce.\sin.\omega-\frac{3}{8}.ce^2.\sin.2\omega}{B}+\frac{3}{4}e^2c^2\frac{1+\sin.2\omega}{B^2}+\&c.\right\},$$

in which ω is an arc whose cosine is $\frac{1}{e}-\frac{1-e^2}{er}$. Hence for every distance SP, we have the angle ASP, and consequently the orbit is determined. When the body comes to the lower apse, $\omega=180°$, and therefore the angle between the apsides

$$=180°.\sqrt{\frac{1-c}{1-4c+ce^2}}.\left\{1-\frac{ce^2}{4(1-4c+ce^2)}+\frac{3e^2c^2}{4.(1-4c+ce^2)^2}+\&c.\right\}$$

The limit of this angle as e diminishes is $180°.\sqrt{\frac{1-c}{1-4c}}$, the very same as given in the second Corollary of Prop. (45.) of Newton.

(224.) This result is applicable to the lunar orbit. Taking $e = ,055$, as in the lunar orbit, the angle between the apsides, computed from the above, will differ only four seconds from the limit, and consequently, the mean motion of the apsides of the lunar orbit, by neglecting the eccentricity, is erroneous, by only eight seconds in a revolution. Observation, however, gives the mean motion of the lunar apsides nearly double that deduced from the limit. To account for this, it is to be observed that the motion of the lunar apsides arises from the disturbing force of the sun, and that the components of this force are,

1°. A force in the direction of the radius vector of the moon's orbit, expressed as a function of this radius vector, and of the angular distance of the sun and moon.

2°. A force perpendicular to the former, expressed also as a function of the same quantities.

The former force, in its mean quantity, is a function of the radius vector *only*; the latter, in its mean quantity, is *cypher*. It has, therefore, been thought that the former force only could influence the motion of the lunar apsides. The result proves this supposition erroneous, however plausible it may appear. Newton certainly computes only that part of the motion arising from the former, and declares it to be half the whole,* which shews that he was fully aware of his not having considered all the disturbing force that could produce this motion. It does not appear possible to compute the mean motion of apsides, except from the general expressions of the forces along the radius vector and tangent.

(225.) There is a case in which the orbit can be determined generally, or in other words, in which equation (*b*) of Prop. (46.) can be completely integrated: this case we shall discuss in the following proposition.

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* "Apsis lunæ est duplo velocior circiter."

PROP. LI.

(226.) *To determine the orbit generally, when the velocity is equal to that from infinity, and the force inversely as the* n^{th} *power of the distance.*

In this case we have $\epsilon^2=1$ in equation (*b*), Prop. (46.) or which is the same thing, $q^2=\frac{2}{n-1}$ in equation (*c*), consequently the function to be integrated is

$$d\omega=\frac{du}{\sqrt{\frac{2\mu u^{n-1}}{(n-1).h^2}-u^2}}=\frac{du}{u.\sqrt{\frac{2\mu u^{n-3}}{(n-1).h^2}-1}}.$$

To integrate this, assume $\frac{2\mu u^{n-3}}{(n-1)h^2}=y^2$, then

$$\frac{(n-3).du}{u}=\frac{2dy}{y},$$

and consequently

$$d\omega=\frac{2}{n-3}\cdot\frac{dy}{y.\sqrt{y^2-1}},$$

let $y=\sec.\,\theta$, then

$$d\theta=\frac{dy}{y.\sqrt{y^2-1}},$$

whence

$$(\omega-a)=\frac{2}{n-3}\cdot\sec.^{-1}y,$$

or restoring for y its value in terms of u, the equation of the orbit is

$$\cos.\frac{n-3}{2}\cdot(\omega-a)=\frac{h.\sqrt{n-1}}{\sqrt{2\mu.u^{\frac{n-3}{2}}}}=\frac{h.\sqrt{n-1}}{\sqrt{2\mu}}.r^{\frac{n-3}{2}}.$$

Let the value, when $\omega=a$, be a, then this equation may be written

$$a^{\frac{n-3}{2}}\cdot\cos.\frac{n-3}{2}(\omega-a)=r^{\frac{n-3}{2}},$$

or if we count ω from the line a,

$$a^{\frac{n-3}{2}}\cdot\cos.\frac{n-3}{2}\cdot\omega=r^{\frac{n-3}{2}}.$$

(227.) We shall enumerate some few examples of this species of curve.

1°. Let $n=2$, and the equation gives us $r=\frac{2a}{1+\cos.\omega}$. This is the focal parabola which has been already amply discussed.

2°. Let $n=3$. In this case the equation fails, and we must revert to the differential equation which gives us the logarithmic spiral as has been already shewn in Art. (172.)

3°. Let $n=4$. The equation gives us $r=\frac{a}{2}\cdot(1+\cos.\omega)$. This is the equation of the cardioide, the diameter of its base being $\frac{a}{2}$. We can readily find the periodic time by integrating the expression $r^2d\omega=hdt$, hence $t=\frac{\pi a^2}{2h}$ for the whole cardioide.

4°. If $n=5$. The equation gives us $r=a.\cos.\omega$. This is the equation of a circle with the pole on the circumference.

5°. If $n=7$. The equation becomes $r^2=a^2.\cos.2\omega$, which is the equation of the Lemniscata, the semiaxis of the generating hyperbola being a. The time and other circumstances of the motion may be found as usual.

(228.) The locus of the points fallen to internally in this class of curves is always of the same nature as the orbit described, for we have in general, by Art. (71.), $\frac{1}{\Delta'^{n-1}}+\frac{1}{\Delta^{n-1}}=\frac{2}{r^{n-1}}$, but in the present instance Δ is infinite, whence $\Delta'=\frac{r}{\sqrt[n-1]{2}}$, and consequently Δ' to r in a given ratio.

(229.) The equation (b) for determining the apsides becomes in this case $\frac{(n-1)h^2}{2\mu}-u^{n-3}=0$, whence evidently there must be an apse when $r^{\frac{n-3}{2}}=\frac{\sqrt{2\mu}}{h.\sqrt{n-1}}$, and in this case $\frac{n-3}{2}\,\omega=0$, or π, and therefore $\omega=0$, or π. If $\frac{n-3}{2}$ be an even number, the orbit has two apsides, and never can have more. The curve is symmetrical on the two sides of an apse, since the cosine of a negative and positive arc is the same.

(230.) If $n>3$: as ω increases, r diminishes, and actually vanishes when $\frac{n-3}{2}\cdot\omega=\frac{\pi}{2}$, or $\omega=\frac{\pi}{n-3}$. In this case the moving point will descend to the centre in a number, $\frac{1}{2n-6}$, of entire revolutions. If $n=3$, it will always require an infinite number of revolutions to cause this descent. In the cardioide this descent will take place in one-half, in the circumferential circle in one-fourth, and in the Lemniscata in one-eight of a revolution.

(231.) If $n<3$: as ω increases, r increases indefinitely, and when $\frac{(3-n)}{2}\cdot\omega=\frac{\pi}{2}$, r is infinite, and the curve parallel to it. In this case the moving point will fly off to infinity in a number of revolutions, $\frac{1}{6-2n}$. If $n=1$, the body will go off to infinity in one-fourth of a revolution, this being an extreme case, and one that can never actually occur. To find whether this infinite branch can ever admit of an asymptote, we find $\frac{1}{p^2}=u^2+\frac{du^2}{d\omega^2}=\frac{2\mu u^{n-1}}{(n-1)h^2}$. This becomes cypher when $r=\infty$, and consequently p becomes infinite, hence the curve has no asymptote.

(232.) If n be *even*, the tangents at the extremities of any chord passing through the centre of force cut at right angles. For $\tan.\theta=-\frac{r\,d\omega}{dr}=\cot.\frac{n-3}{2}\cdot\omega$, and $\tan.\theta'=-\cot.\left((n-3).\frac{\pi}{2}+\frac{n-3}{2}\cdot\omega\right)=\tan.\frac{n-3}{2}\cdot\omega$, where θ and θ' are the angles made by the chord with the tangents. Hence $\tan.\theta.\tan.\theta'=1$, and consequently $\theta+\theta'=\frac{\pi}{2}$. The reader will instantly recognise the truth of this elegant property in the focal parabola and the cardioide. If n be *odd*, we have $\tan\theta.\cot.\theta'=1$, and consequently $\theta=\theta'$. This is obvious in the two instances of the circumferential circle and the logarithmic spiral.

(233.) It is obvious that in the case which we at present consider, the orbit never can have more than one apsidal distance.

SECTION VI.

ON THE DISCOVERY OF UNIVERSAL GRAVITATION.—THE MASSES OF THE PLANETS, AND PROBLEM OF TWO BODIES.—THE ATTRACTIONS OF SOLIDS COMPUTED, AND MASSES OF THE PLANETS DETERMINED.

§ I.

(234.) Immediately after the discovery of the true system of the sun and planets by the celebrated Copernicus, all the most noted philosophers of Europe became zealously engaged in prosecuting this discovery, and in endeavouring more minutely to ascertain the nature and laws of the planetary motions. One of the most industrious and successful in this field was Kepler; aided by the advice and observations of Tycho Brahe, this celebrated man, after many failures and conjectures, succeeded in establishing three of the most important discoveries that has ever been made in physical science. They are,

1°. *The planets move round the sun, describing equal areas in equal times.*

2°. *They describe ellipses round the sun, situated in the focus of each ellipse.*

3°. *The squares of the periodic times are as the cubes of the mean distances from the sun.*

For the manner of establishing these facts, and the consequences immediately deducible from them, we refer to the notes. In the sequel of this section we shall see the important consequences which result from them, when combined with Newton's laws of motion. These discoveries, not as those of former philosophers, founded on, and terminating in mere appearances, were readily seen to be intimately connected with the forces and laws of force which actuate the bodies of the solar system. They must also, if it were only by suggesting the inquiry, have been highly instrumental in leading to the discovery of those forces. Accordingly we find that the question was soon entered upon, and the first attempt, in this way, was made by Kepler himself. Notwithstanding some vain speculations as to the motive cause of the planets, he was frequently led to many sound views on the subject of universal gravitation.

" Gravity," says he, " is no other than a mutual affection of bodies by which they tend to unite."

" Two isolated bodies would run together like two magnets, passing over spaces reciprocally as their masses. If the moon and earth, for instance, were not kept asunder by some animal or other equivalent force, they would run together, the moon describing $\frac{53}{54}$ of the way, supposing them equally dense."

" If the earth ceased to attract the waters of the ocean, they would fall to the moon by its attraction."

" This force of the moon extending to the earth produces the flux and reflux of the sea."

These remarks contain the first germ of that celestial mechanism which has been so successfully developed by Newton and his successors. The day was not yet arrived for such a revolution in physical science as was afterwards effected. Greater discoveries in Dynamics, and a more powerful calculus were required, before any such object could be obtained. These sciences were beginning at this time to advance considerably. Galileo, by his numerous mechanical discoveries, and more particularly by his ascertaining the path of a projectile at the earth's surface, had already contributed powerfully to the advancement of Dynamics; whilst Fermat, by his method of *maxima* and *minima*, and Napier, by the invention of Logarithms, and his discoveries in Trigonometry, laid the foundation of the calculus.

(235.) The theorems of Huyghens, concerning uniform circular motion, were the most important accession which Physical Science received in the period between Kepler and Newton. Geometry, Mechanics, and Optics are all indebted to Huyghens for a great number of discoveries. His application of the pendulum to clocks, his theory of evolutes, and theorems on centrifugal force have establised his fame. If he had but thought of combining his theorems on centrifugal force with his theory of evolutes and laws of Kepler, he would probably have anticipated Newton in a theory of curvilinear motion, and the discovery of universal gravitation. We shall see what might have been inferred from combining these theorems with Kepler's laws. By the theorems of Huyghens $f \propto \frac{v^2}{r}$, (supposing the planets to move in circular orbits, which supposition differs but little from the truth), and $v \propto \frac{r}{T}$, whence $f \propto \frac{r}{T^2}$; but by Kepler's third law, $T^2 \propto r^3$, hence $f \propto \frac{1}{r^2}$. Thus we see how the force would vary from one orbit to another. The hypothesis that $T^2 \propto r^3$, is certainly not exact, as Kepler only shewed this to be the case for elliptical orbits; but as the law appeared independant of the eccentricities, it was natural to suppose it would subsist even though the orbits became circles.

(236). As this law thus appears to hold from one orbit to another, analogy would lead us to conjecture that it may also hold in different parts of the same orbit. Let us examine what infereuce in this respect might be drawn. By Kepler's second law, the orbits described are focal ellipses, and by the first, the velocities at the apsides are inversely as the distances. But since the curvatures at the apsides are the same, the centrifugal forces, by Huyghens's theorems, are as the squares of the velocities, and therefore inversely as the squares of the distances. Thus the discoveries of Huyghens combined with those of Kepler would have sufficed for discovering the law of gravitation; for a law which holds true from one planet to another, and which can be verified for each planet in particular at its perihelion and aphelion, must hold true for all points of the planetary orbits, and generally for all distances from the sun. In order to establish this point in an incontestible manner, it would be requisite to have an expression for the force which, directed to the focus of an ellipse, would cause a body to describe this curve. This great discovery was reserved for Newton: he, proceeding merely from the principles of terrestrial mechanics, discovered the force and laws of force requisite for the description of any curve about a single centre of force; he shewed as a consequence of the force being single, that equal areas would be described in equal times about the centre to which this force is directed, that if the orbit were a focal ellipse, the law of force would be the inverse square of the distance, and moreover, that gravitation to the centre of force ought not to vary from one revolving body about the same centre to another, except in consequence of a change of distance. The laws of Kepler enabled Newton to apply these general principles to the planetary system as follows:—

1°. From the proportionality of areas to the times, it follows that the force which solicits each planet must be directed to the sun's centre. This readily appears from Art. (45.), for in order that there should be an equable description of areas, we must have $Xy—Yx=0$.

2°. The orbits being ellipses with the sun in the focus, proves that the force is inversely as the square of the distance. This has been shewn in Prop. (31.), where for this orbit we found $F=\frac{h^2}{a(1-e^2)}\cdot\frac{1}{r^2}\cdot$ The converse of this has been shewn in Prop. (40.), and may also be shewn as follows; let the force be $\frac{\mu}{r^2}$, μ being a constant coefficient, then the equation

$$Fdr=h^2.\left(\frac{dr}{r^3}-d.\frac{dr^2}{2r^4d\omega^2}\right)$$

will be satisfied by the equation

$$r=\frac{a.(1-e^2)}{1+e\cos.(\omega-\pi)},$$

when F is changed into $\frac{\mu}{r^2}\cdot$ Now in order actually to determine this ellipse, we have to determine three constants, a, e, and π; the equation $\mu=\frac{h^2}{a(1-e^2)}$ reduces this number to two, and the differential equation between r and ω being of the second order, enables us to determine these two. Consequently the equation of conic sections is the complete integral of this. This is substantially the same as the solution given by Newton, Cor. (1.), Prop. (13.), and proves the mistake of Bernouilli, when he said that Newton had not solved the inverse problem of central forces.

3°. From the third of Kepler's laws we may infer that all bodies gravitate equally towards the sun at equal distances. In order to this, we need but determine the intensity of the force F. This intensity depends on the coefficient $\frac{h^2}{a(1-e^2)}\cdot$ Let t denote the periodic time in the ellipse, then by the first of Kepler's laws,

$$\tfrac{1}{2}h\,dt\ :\ \pi a^2\sqrt{1-e^2}\ ::\ dt\ :\ t,$$

whence

$$\frac{h^2}{a(1-e^2)}=\frac{4\pi^2.a^3}{t^2};$$

but by Kepler's third law, $t^2=k^2.a^3$, therefore

$$\frac{h^2}{a(1-e^2)}=\frac{4\pi^2}{k^2}$$

a constant quantity.

To extend this to the comets:—by the preceding, $h=\frac{2\pi\sqrt{a(1-e^2)}}{k}$ as a consequence of Kepler's third law, and therefore the areas are as the square roots of the parameters of the described orbits; now if we suppose the comets to describe parabolic orbits, observation proves the areas to be as the square roots of the parameters, as well as for the planets. Hence, denoting the perihelion distance by D, we may write $h=\frac{2\pi\sqrt{2D}}{k}$, and therefore we have

$$\frac{2\pi\sqrt{2D}}{2k}\cdot dt : \pi a^{\frac{3}{2}}\sqrt{2D} :: dt : t,$$

whence

$$t^2=k^2a^3;$$

and consequently Kepler's law is equally true for the comets. Hence we can determine the greater axes if we know the periodic times. Since in general we have $F=\frac{4\pi^2}{k^2}\cdot\frac{1}{r^2}$, it follows that the force F is inversely as the square of the distance from the sun, and that it only varies from one planet to another in consequence of a change of distance.

(237.) We are thus, by combining the laws of Kepler with the discoveries of Newton, led to consider the sun as the focus of an attractive force, which extends indefinitely in every direction, and decreases as the square of the distance increases. The law of areas shews us that this force is constantly directed to the sun's centre, the elliptical form of the orbits proves that the force is inversely as the square of the distance, and the law of the periodic times proves that this force is the same at equal distances for all bodies. If from the planets we pass to the satellites, their near observance of Kepler's laws proves that they gravitate towards their respective primaries with forces inversely as the squares of the distances. They must also gravitate very nearly as their primaries towards the sun, in order that their relative motions about their primaries may be nearly the same as if these planets were at rest. It is difficult to determine the law of gravitation for the satellites by the second of Kepler's laws, from the difficulty of determining the elliptical form of the orbits, and whether the primary is situated in the focus of each ellipse. From the third of Kepler's laws, which observation proves to be observed by the satellites of each system, we can readily infer the law of gravitation; for the orbits being nearly circular, we have the forces directly as the radii, and inversely as the squares of the times, but the squares of the times are as the cubes of the distances, therefore, &c.

The frequent recurrence of the eclipses of the satellites enables us easily to determine their synodic revolutions, and thence knowing the motion of the primary to determine their periodic times.

(238.) As the earth has but one satellite, the ellipticity of the lunar orbit is the only phænomenon which can indicate to us the law of force by which the moon tends to the earth. This ellipticity, however, is so much disturbed by the action of the sun and planets, that some doubt may exist as to the law of force. Analogy indeed would lead us to conclude that the law is the same as for the other satellites; but the following method, depending on experiments instituted on terrestrial gravity, places the matter beyond a doubt. This method consists in determining the lunar parallax by means of experiments on the length of the seconds pendulum, and comparing it with the same determined by observations in the heavens. On the parallel,* of which the square of the sine of the latitude is $\frac{1}{3}$, the pendulum gives the space fallen through in a second 3,65548 metres; the centrifugal force on this parallel is the 432^d part of the force of gravity, therefore, the space due to the action of the earth is $3^{\text{m}}{,}65548.(1+\frac{1}{432})$; this is to be diminished in the proportion of the square of the earth's rad. to the square of the moon's distance, or in other words, to be multiplied by the square of the sine of the parallax, to obtain the corresponding space at the moon; hence the space fallen by the moon in consequence of the earth's attraction is $\sin.^2 p.\, 3^{\text{m}}{,}65548\,(1+\frac{1}{432})$, p denoting the parallax. But the moon does not actually fall so much, the action of the sun prevents it, and diminishes it by its 358th part in its mean quantity, hence the above expression must be multiplied by $(1-\frac{1}{358})$. Again, this space must be augmented in the ratio of the sum of the masses of the moon and earth to the mass of the earth, as shall be shewn further on; this ratio is expressed by the fraction $\frac{59{,}7}{58{,}7}$, as deduced from the phænomena of the tides, hence the actual approach of the moon and earth is

$$3^{\text{m}}{,}65548.\,(1+\tfrac{1}{432})\,(1+\tfrac{1}{358}).\,\tfrac{59{,}7}{58{,}7}\,.\sin.^2 p.$$

Let a represent the mean distance of the moon from the earth, and t the duration of a sidereal revolution expressed in seconds, then $\frac{2a\pi^2}{t^2}$ will be the versed sine of the arc described in a second, or the space fallen through towards the earth. This value of a is equal to the radius of the earth on the above-mentioned parallel divided by $\sin. p$, this radius is 6369514^{m}, therefore $a = \frac{6369514}{\sin. p}$; substituting this value for a; for π the fraction $\frac{355}{113}$; and for t, $2732166''$; the space descended is $\frac{2.(355)^2.6369514^{m}}{(113)^2 \sin. p (2732166)^2}$. By equating this with the above we find $\sin.^3 p$, and from this, p, which comes out $10536'',2$. The parallax to be assumed in the above computation must be assumed as the mean parallax, independant of the lunar inequalities; this is called the *constant part of the parallax*. By observation this parallax is 10540,7, which differs but little from the above; hence the *principal* force which retains the moon in her orbit is terrestrial gravity diminished in the ratio of the square of the distance. The distance in this case is to be reckoned from the centre of gravity of the earth, it must therefore for the sun and planets be counted from their centres of gravity, since the attractive force for them is of the same nature.

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* The reason of selecting this parallel shall be shewn when we come to treat on 'Attractions'.

(239.) The sun and the planets which have satellites are in consequence endowed with an attractive force, which, diminishing as the square of the distance increases to infinity, comprises in the sphere of its activity all bodies. Analogy would lead us to infer that a similar force resides in all the planets and comets, but we may be directly assured of it as follows. It is a constant law of nature that one body cannot act upon another without an equal and contrary reaction; thus the planets and comets being attracted towards the sun ought to attract it again according to the same law. The satellites for the same reason attract their primaries; this attractive property is then common to the planets, the comets, and the satellites, and consequently we may regard the gravitation of the celestial bodies, one towards the other, as a general property of the universe. Any the slightest deviation from the law of the inverse square of the distance would be clearly perceptible on the motions of the *perihelia* of the planetary orbits,* on which observation has enabled us to perceive no other than imperceptible motions, of which we shall develope the cause.

(240.) *All particles of matter attract mutually with forces directly as their masses, and inversely as the squares of their distances.*—This law of nature has been announced in the preceding pages, and the second part of it proved by a copious induction. Nothing has as yet been said of the manner in which this force depends on the masses of the attracting bodies. That which retains the planets in their orbits is the resultant of the attractions of all the molecules of the sun on the molecules of each planet; from the smallness of the dimensions of the sun and planets relative to their mutual distances, these forces may be looked upon as equal and parallel; their resultant is consequently equal to their sum, and we may conclude, that the distance remaining the same, the motive force of each planet is proportional to the product of its mass by that of the sun.

In order to express the intensity of this force let us select some unit of distance, and also an unit of mass and of time, and let us take as the unit of force f, that constant accelerating force which produces in the unit of time a velocity equal to the unit of length. Then the motive force of any planet is Mmf at the unit of distance, M and m denoting the masses of the sun and planet respectively, and $\frac{Mmf}{r^2}$ for any distance r. The coefficient, f, depends on the nature of matter, there is no reason to believe it mutable with the time.

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* *Vide* Appendix.

(241.) Action and reaction being equal and contrary, there is an equal motive force acting on the sun. They would, consequently, if left to their mutual action, come together, having passed over spaces inversely as their masses. In general, if the planet be projected in any direction, and that we propose to determine its apparent motion round the sun regarded as a fixed point, we must conceive applied to the sun at each instant a velocity equal and contrary to that which it receives from the planet; this is the same as applying to the planet a motive force directed to the sun and equal to $\frac{\mathrm{M}mf}{r^2}\cdot\frac{\mathrm{M}}{m}$, or to $\frac{m^2f}{r^2}$; consequently, in the motion in question, the motive force of the planet m will be constantly directed to the sun, and equal to $\frac{\mathrm{M}mf}{r^2}+\frac{m^2f}{r^2}$; and therefore by dividing by the mass m, the accelerating force $\frac{\mu}{r^2}$ will be expressed by $\frac{(\mathrm{M}+m)f}{r^2}$.

This value ought to be substituted for μ in the equations of elliptical motion that have been given in the preceding pages, in order to render them applicable to a system of two *bodies* not mere points. For instance, in Art. (94.) the equation between t and a ought to be written $\frac{t^2}{a^3}=\frac{4\pi^2}{(\mathrm{M}+m)f}$. The relation of t^2 to a^3, which depends thus on the quantity m, will not be the same for two planets of which the masses are unequal. Hence we may perceive that Kepler's third law is not rigorously exact. The observations, however, which lead to this law, prove that if not exact it must be nearly so, and that consequently the masses of the planets must be small relative to that of the sun; and in fact we find that the mass of Jupiter, the largest of all the planets, is not a thousandth part of the sun. Our next object would be to shew how to compute the masses and densities of the bodies of the solar system, it is previously necessary for this, as well as for other reasons, to digress briefly on the subject of Attractions.

§ II.

(242.) The calculation of the attractions exercised by bodies of different forms is of the highest importance in the system of the world. It is from it we can infer the justice of the assumption, that the joint action of the several particles of a planet is the same as if all these particles were concentrated in a single point. The determinations of the masses of the planets, and of their figures, also require this previous calculation.

PROP. LII.

To compute the attraction of a spherical shell or globe on a particle at a given distance from the centre, the force varying directly as the distance.

(243.) Let r be the distance of the point attracted from the centre of any spherical shell which constitutes part of the globe, let u be the radius of this shell, and du its thickness. Let θ be the angle which the radius u makes with the right line r, ω the angle which the plane of the right lines r and u makes with a fixed plane passing through the lines, then the element of the spherical shell is $u^2 . du . d\omega . d\theta . \sin . \theta$. Let δ be the distance of this attracting element from the attracted point, ρ its density, and f the attracting force of the unit of mass at the unit of distance, then the attraction of the element in its own direction will be $\rho . fu^2 . du . d\omega . d\theta . \sin . \theta . \delta$. The component of this attraction tending to the centre is found by multiplying by the cosine of the angle between δ and r, viz. $\frac{r-u.\cos.\theta}{\delta}$. Hence the attraction towards the centre is $\rho.f.u^2du.\,d\omega.\,d\theta.\sin.\theta.\,(r-u\cos.\theta)$. Denoting the attraction of a shell of any thickness by A, we have from this

$$A=\iiint\rho.f.u^2.\,du.\,d\omega.\,d\theta,\sin.\theta.\,(r-u\cos.\theta).$$

Integrating, relative to ω, from $\omega=0$ to $\omega=2\pi$, we find for the attraction of a *ring* of indefinite thickness,

$$A=2\pi\rho.f\iint u^2du.\,d\theta.\sin.\theta.\,(r-u\cos.\theta).$$

Integrating next on the hypothesis that u is constant between the limits $\theta=0$ and $\theta=\pi$, we find the attraction of a shell of indefinite thickness,

$$A=4\pi\rho.fr.\int u^2du.$$

Integrating lastly relative to u, from $u=l$ to $u=l'$, we find

$$A=\frac{4\pi\rho.f.r}{3}\cdot(l'^3-l'^3)$$

Let M denote the mass of the shell, then $M=\frac{4\pi\rho}{3}\cdot(l'^3-l^3)$, whence we have

$$A=M.f.r.$$

(244.) From this we may perceive that the attraction of the shell for this law of force is precisely the same as if the entire mass of the shell were concentrated in its centre.

PROP. LIII.

(245.) *To compute the attraction of a spherical shell or globe on a particle, the force varying inversely as the square of the distance.*

Using precisely the same notation as in the preceding proposition, we have for the attraction of the element of the solid decomposed in the direction of the centre, $\rho . f u^2 du . d\omega . d\theta . \sin . \theta . \frac{r - u \cos . \theta}{\delta^3}$. Hence

$$A = \iiint \rho . f . u^2 du . d\omega . d\theta . \sin . \theta . \frac{r - u \cos . \theta}{\delta^3} .$$

Integrating relative to ω, from $\omega = 0$ to $\omega = 2\pi$, we find for the attraction of a *ring*,

$$A = 2\pi \rho f . \iint u^2 du . d\theta . \sin . \theta . \frac{(r - u \cos . \theta)}{\delta^3} .$$

This equation is to be integrated relative to θ, from $\theta = 0$ to $\theta = \pi$; or eliminating θ by the equation

$$\delta^2 = r^2 - 2ru \cos . \theta + u^2 ,$$

it is to be integrated relative to δ between the limits $(r + u)$ and $(r - u)$, if the point attracted be without the shell. Performing this integration, we find

$$A = \frac{4\pi \rho . f .}{r^2} . \int u^2 du .$$

Integrating this between the limits l' and l, we find

$$A = \frac{4\pi \rho . f (l'^3 - l^3)}{3 r^2} = \frac{M . f}{r^2} .$$

Hence in this case, as in the preceding, we find that a shell or globe attracts a point situated without it, as if its entire mass were collected into the centre.

(246.) In this case, if the point attracted were within the shell, we should integrate relative to δ between the limits $(u+r)$ and $(u-r)$, and the result obtained would be cypher.—*Vide* Part I., Sect. XII., Prop. (70). This is not the case for the preceding law of force. In that case the shell always attracts, whether the point attracted be within or without, as if the entire mass of the shell were collected in the centre.

The property of attracting, as if the entire mass were collected in the centre, is very remarkable, and it is interesting to determine whether it holds for other laws of attraction than the two preceding.

PROP. LIV.

(247.) *To determine for what laws of force a spherical shell or globe attracts, as if its entire mass were collected in its centre.*

In the first place, we may observe if the law of attraction be such that a homogeneous sphere attracts a point placed outside it, as if the entire mass were collected in its centre; the same law must hold for an indefinitely thin shell of the thickness du. Representing the law of force by $\phi(\delta)$, the action of the element of such a shell decomposed in the direction of the centre will be proportional to

$$u^2 du.\, d\omega.\, \sin.\theta\, d\theta. \frac{r-u.\cos.\theta}{\delta}.\, \phi(\delta),$$

and since we have $\frac{r-u.\cos.\theta}{\delta}=\left(\frac{d\delta}{dr}\right)$, this becomes

$$u^2 du. d\omega. d\theta. \sin.\theta. \left(\frac{d\delta}{dr}\right). \phi\delta.$$

Now if we designate $\int d\delta.\phi(\delta)$ by $\phi_{,}(\delta)$, we shall have the total attraction of the spherical shell on the point attracted by means of the integral $u^2 du \int d\omega. d\theta. \sin.\theta. \phi_{,}(\delta)$ differentiated relative to r, and divided by dr.

Integrating this relative to ω between the limits 0 and 2π, we have

$$2\pi u^2 du \int d\theta. \sin.\theta.\phi_{,}(\delta).$$

Differentiating the value of δ relative to θ, we have $d\theta.\sin.\theta = \frac{\delta.d\delta}{ru}$, whence the expression for the attraction becomes

$$2\pi\frac{udu}{r}.\int\delta.d\delta.\phi_{,}(\delta).$$

This integral being taken between the limits $(r-u)$ and $(r+u)$, and being of the form $\psi(\delta)$, we have

$$\frac{2\pi udu}{r}.\{\psi(r+u)-\psi(r-u)\}.$$

Now to determine $\phi(\delta)$, on the condition that the attraction of the shell should be the same as if collected in the centre. The mass of the shell is $4\theta u^2 du$, and its attraction $4\pi u^2 du\phi(r)$, we have then

$$2\pi udu. \frac{d.\left\{\frac{1}{r}(\psi(r+u)-\psi(r-u))\right\}}{dr} = 4\pi u^2 du\phi(r).$$

Dividing, and integrating relative to r, we have

$$\psi(r+u)-\psi(r-u)=2ru\int dr.\phi(r)+r\mathrm{U}.$$

U being a function of u and of constants. Representing the left hand member of this equation by R, we find

$$\frac{d^2\mathrm{R}}{d^2r}=4u\phi(r)+2ru.\frac{d.\phi(r)}{dr},$$

$$\frac{d^2\mathrm{R}}{du^2}=r.\frac{d^2\mathrm{U}}{du^2}.$$

But we have $\frac{d^2\mathrm{R}}{dr^2}=\frac{d^2\mathrm{R}}{du^2}$ by the nature of the function R, hence

$$\frac{2\phi(r)}{r}+\frac{d\phi(r)}{dr}=\frac{1}{2u}.\frac{d^2\mathrm{U}}{du^2}.$$

One member of this equation being independant of r, and the other of u, each ought to be equal to an arbitrary constant, which we shall designate by 3 A, we have then

$$\frac{2\phi(r)}{r}+\frac{d.\phi(r)}{dr}=3\mathrm{A},$$

whence by integrating we obtain

$$\phi(r)=\mathrm{A}r+\frac{\mathrm{B}}{r^2}.$$

This result shews us that all the laws of attraction for which a sphere acts as if it were collected in its centre, are either the direct law of the distance, the inverse square of the distance, or laws compounded of these.

If $\mathrm{A}=0$, we shall have the law of nature, and in fact we may see that in the infinite number of laws of force that give the attraction nothing at infinite distances, that of nature is the only one in which spheres possess this singular property.

PROP. LV.

(248.) *To determine the law of force when a shell attracts a point placed within it equally in all directions.*

It follows from the preceding analysis that the attraction of a spherical shell, of the thickness du, on a point placed within it, has for it expression,

$$2\pi u du. \frac{d. \left\{ \frac{1}{r}(\psi(u+r)-\psi(u-r)) \right\}}{dr}.$$

In order that this expression may vanish, we have

$$\psi(u+r)-\psi(u-r)=r.\mathrm{U},$$

it being a function of u independant of r. Differentiating this twice in order relative to r, we have

$$\frac{d^2\psi(u+r)}{dr^2}-\frac{d^2\psi(u-r)}{dr^2}=0.$$

This equation being true, whatever be the values of u and r, it follows that $\frac{d^2\psi(\delta)}{d\delta^2}$ ought to be a constant quantity for every value of δ, and therefore that $\frac{d^3\psi(\delta)}{d\delta^3}=0$: we have from the analysis in the preceding proposition,

$$\frac{d.\psi(\delta)}{d\delta}=\delta.\phi_{\prime}(\delta),$$

whence, by twice differentiating, we have

$$2\phi(\delta)+\delta.\frac{d\phi(\delta)}{d\delta}=0.$$

Multiplying each term by δ, and integrating, we have

$$\phi(\delta)=\frac{B}{\delta^2},$$

which is the law of nature.

PROP. LVI.

(249.) *To find the attraction of a circular plate on a point in an axis perpendicular to its plane from the centre.*

Let x be the distance of the point in the axis from the centre, z the distance of any element of the circular plate from the centre, r the distance of this element from the point attracted, and y the radius of the circle. Then the element of the circle is $zd\omega.dz$, and its attraction on the point decomposed in the direction of the centre is $\frac{\rho f.zd\omega.dz.x}{r^3}$. Integrating this from $\omega=0$ to $\omega=2\pi$, we find for the attraction of a ring $\frac{2\pi\rho fx.zdz}{r^3}$; writing for r its value $\sqrt{x^2+z^2}$, and integrating relative to z from $z=0$ to $z=y$, we find for the attraction of the entire plate $2\pi\rho f.\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)$.

PROP. LVII.

(250.) *To find the attraction of an oblate spheroid on a point in its axis or axis produced.*

By the preceding, the attraction of a circular plate of indefinite thickness perpendicular to the axis of rotation is $2\pi\rho f\left(1-\frac{x}{\sqrt{x^2+y^2}}\right)dx$. In the present case $y^2=\frac{a^2}{b^2}\cdot(2bx-x^2)$, a denoting the equatoreal, and b the polar radii, substituting which we have

$$dA=2\pi\rho f.\left\{dx-\frac{bxdx}{\sqrt{2a^2bx-x^2}}\right\}$$

$$2\pi\rho f.\left\{dx+\frac{b(a^2b-c^2x)dx}{c^2.\sqrt{2a^2bx-x^2}}-\frac{a^2b^2dx}{c^2.\sqrt{2a^2bx-x^2}}\right\}.$$

Hence integrating the several terms,

$$A=2\pi\rho f.\left\{x+\frac{b}{c^2}\cdot\sqrt{2a^2bx-c^2x^2}-\right.$$

$$\left.\frac{2a^2b^2}{c^3}\cdot\tan.^{-1}.\sqrt{\frac{c^2x}{2a^2b-c^2x}}\right\}+k.$$

Taking this between the limits $x=0$ and $x=2b$, we find $k=0$, and

$$A=4\pi\rho f.b\left\{\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\cdot\tan.^{-1}.\frac{e}{\sqrt{1-e^2}}\right\}\qquad(a)$$

This result is obtained by the method suggested by Newton, Sect. XIII., Prop. (90). The methods of obtain-ing the same, used by Clairaut, *Fig. de la Terre*, p. 179, and by Airy in his *Tracts*, p. 66, are much more tedious and intricate.

(251.) Substituting for the arc in formula (*a*), the series in powers of the tangent, (*see Trig.* p. 106), we can readily obtain the following series, substituting for *e* its value

$$\mathrm{A} = 4\pi\rho f b.\left\{\frac{a^2}{3b^2} - \frac{c^2a^2}{5.6^4} + \frac{c^4a^2}{7.6^6} - \frac{c^6a^2}{9.6^8} + \&\text{c.}\right\}.$$

Now let $a = b.(1+\delta)$, and consequently $c^2 = b^2.(2\delta+\delta^2)$, then

$$\mathrm{A} = 4\pi\rho f.b.\left\{\frac{(1+\delta)^2}{3} - \frac{(2\delta+\delta^2).(1+\delta)^2}{5} + \right.$$

$$\left.\frac{(2\delta+\delta^2).^2(1+\delta)^2}{7} - \&\text{c.}\right\}.$$

This being arranged in powers of δ, becomes

$$\mathrm{A} = \frac{4\pi\rho.fb}{3}.\{1 + \tfrac{4}{5}.\delta - \tfrac{2}{7}.\delta^2 + \tfrac{8}{105}.\delta^3, \&\text{c.}\}$$

If the spheroid differ but little from a sphere, the two first terms of this series would suffice.—*See* Airy's *Tracts*, p. 66: Robinson's *Mech. Phil.* vol. i., p. 242.

LEMMA.

(252.) *To find the attraction of a pyramid whose base is* X, *and altitude* r, *on a particle at the vertex, the base being infinitely small and perpendicular to* r.

Let Z be a section parallel to the base at a distance, z, from the vertex, then the attraction of $\mathrm{Z}dx$ is $\frac{\mathrm{Z}f.dz}{z^2}$, but

$Z : X :: z^2 : r^2$, therefore the attraction of Zdx is $\frac{X.f.dz}{r^2}$, this integrated between the limits $z=0$ and $z=r$, gives for the attraction of the pyramid $\frac{Xf}{r}$.

PROP. LVIII.

(253.) *To find the attraction of an oblate spheroid on a point in its equator.*

Conceive an elliptical meridian with a tangent thereto at the equator, let the radius of the equator be a, and the polar radius b; then the equation of the meridian is

$$r=\frac{2b^2a.\cos.\theta}{c^2.\sin.^2\theta+b^2},$$

θ being the angle between r and the equatoreal radius. Let the meridional plane be conceived to revolve on its tangent through an indefinitely small angle, $d\omega$, and the wedge generated has for its element a pyramid, whose length is r, and whose base is $r^2.d\omega.d\theta.\cos.\theta$, whence its attraction, by the lemma, is $rfd\omega.d\theta.\cos.\theta$; this decomposed in the direction of the centre gives us the differential of the attraction, A, of the wedge in this direction. Hence, $dA=r.f.d\omega.d.\sin.\theta.\cos.\theta$. Substituting for r from the equation of the curve,

$$dA=\frac{2b^2a\rho fd\omega.d.\sin.\theta\,(1-\sin.^2\theta)}{c^2.\sin.^2\theta+b^2},$$

or writing x for $\sin.\theta$, this may be written

$$dA=2b^2a\rho f.d\omega.\left\{\frac{c^2+b^2}{c^2}.\frac{dx}{c^2x^2+b^2}-\frac{dx}{c^2}\right\}.$$

Hence

$$A=2b^2a.f.d\omega.\left\{\frac{1}{e^2}\cdot\frac{1}{bc}\cdot\tan.^{-1}.\frac{cx}{b}-\frac{x}{c^2}\right\}+k.$$

Taking this between the limits $x=0$ and $x=1$, we obtain for the semi-ellipse

$$A=2b^2a\rho.f.d\omega\left\{\frac{1}{e^2}\cdot\frac{1}{bc}\cdot\tan.^{-1}.\frac{c}{b}-\frac{1}{c^2}\right\} \qquad (b)$$

Let us conceive a circle in the meridional plane, whose diameter is $2b$, and which touches the meridian at the point attracted; for the wedge generated by it we have, by the above,

$$dA=2b.\rho.f.d\omega.dx.(1-x^2).$$

This integrated, and taken between the same limits, gives

$$\frac{4bf.d\omega}{3};$$

by comparing which with equation (a), we obtain the ratio of the attractions of the elliptic and circular wedges; the same expresses the ratio of the attractions of the whole ellipsoid and globe whose diameter is $2b$, but the attraction of the globe is

$$\frac{4\pi f.b}{3}.$$

Hence the attraction E of the spheroid is

$$2b.\rho.f.\pi\left\{\frac{1}{e^2}\cdot\frac{a}{c}\cdot\tan.^{-1}.\frac{c}{b}-\frac{ab}{c^2}\right\},$$

or

$$E=2b\rho f.\pi.\left\{\frac{1}{e^3}\cdot\tan.^{-1}.\frac{e}{\sqrt{1-e^2}}-\frac{\sqrt{1-e^2}}{e^2}\right\} \qquad (c)$$

See AIRY's *Tracts*, p. 69, where the same result is obtained by a most circuitous process.

(254.) By treating series (c) as series (a) has been treated in the preceding proposition, we can readily obtain

$$E=\frac{4b\rho f.\pi}{3}.\left\{1+\tfrac{2}{5}.\delta-\tfrac{9}{35}.\delta^2+\tfrac{11}{105}.\delta^3, \&c.\right\}$$

If the spheroid differ but little from a sphere, we need take only the two first terms of this series.

PROP. LIX.

(255.) *If there be two similar and similarly posited concentrical spheroids, and if a tangent be drawn at the pole of the inner one to meet the other, the attraction of the inner one on a corpuscle at its pole is equal to the attraction of the whole spheroid, on a particle at the extremity of the tangent, parallel to the axis.*

Let us suppose both spheroids divided into wedges by planes intersecting at any small angles in the tangent. Again, let these planes be divided into pairs of pyramids equally inclined to the axis of the inner spheroid, and terminating at its pole; also, into other pairs respectively parallel to these terminating at the point on the surface. Now by a property of the conic sections, the sum of the lengths of one of the former pairs is equal to the sum of the lengths of one of the latter pairs, if they lie all at the same side of the tangent; and the sum of the former pair is equal to the difference of the latter pair, if one of the latter should lie on the contrary side. Hence, since the attractions of similar pyramids are as their lengths, the sum of the attractions of each pair of pyramids is the same in either case, and consequently the attractions of the entire wedges are the same on each point, these attractions being estimated along lines perpendicular to the greater axes of those wedges.

Again, the attractions of each of those wedges been decomposed into two, one perpendicular to the plane of the central wedge, and the other in that plane; the former attractions will mutually destroy each others effects, and the latter will still continue equal.

(256.) After the same manner it may be demonstrated that if a tangent be drawn perpendicular to the plane of the equator of the inner spheroid, the attraction of the inner one on a corpuscle at the point of contact is equal to the attraction on a corpuscle on the outer surface parallel to the plane of its equator.

PROP. LX.

(257.) *To find the attraction on a point on the surface in a direction parallel to the axis.*

This attraction is the same as the attraction of a spheroid on a point at its pole, whose polar semi-axis is equal to the distance from the plane of the equator of the point attracted by the spheroid. Let this distance be *m*, then by formula (*a*), Prop. (57.), this attraction is

$$4\pi\rho.f.m.\left\{\frac{1}{e^2}-\frac{\sqrt{1-e^2}}{e^3}\cdot\tan.^{-1}.\frac{e}{\sqrt{1-e^2}}\right\}.$$

(258.) Similarly the attraction in a direction perpendicular to the axis is by formula (*c*), Prop. (58.),

$$4\pi\rho.f.m.\left\{\frac{1}{e^3}\cdot\tan.^{-1}.\frac{e}{\sqrt{1-e^2}}-\frac{\sqrt{1-e^2}}{e^2}\right\}.$$

In which expression m denotes the same as in the last Art. If n denote the distance from the axis, this becomes

$$4\pi\rho . f. n. \left\{ \frac{\sqrt{1-e^2}}{e^3} \cdot \tan.^{-1}. \frac{e}{\sqrt{1-e^2}} - \frac{1-e^2}{e^2} \right\}.$$

PROP. LXI.

(259.) *The attraction of a shell bounded by similar and concentric spheroidal surfaces on a point within it is cypher.*

The attractions of the whole pyramids terminating at the point attracted would be as their lengths; also, the attraction of the pyramids terminated by the point attracted, and reaching to the inner surfaces would be as their lengths; but these latter pyramids are wanted, therefore the attractions of the remaining *frusta* are as their lengths, but these lengths are equal, and consequently the attractions are equal in opposite directions.

(260.) Hence the attractions of a spheroid on a point within it, both in directions perpendicular and parallel to the axis, can be found by the preceding proposition.

PROP. LXII.

(261.) *In an oblate spheroid the attraction on any point on the surface is proportional to the normal.*

By Cor. III., Prop. (91.), of the *Principia*, if a line be drawn from any point of the surface to the centre, the attractions towards the centre of corpuscles placed on any points of this line would be inversely as their distances from the centre, also, the centrifugal forces decomposed in the same direction would be in the same ratio; consequently the whole central forces would be in this ratio. But since the spheroid is in equilibrio, the weights of the columns terminating at the centre from all points of the surface must be the same; consequently the superficial gravities towards the centre must be inversely as the distances, but these are only the components of the normal gravities which are consequently to each other inversely as the perpendiculars on the conjugate diameters, that is, by the properties of the conic sections directly as the normals.

(262.) Hence denoting the equatoreal gravity by E, and the gravity in any latitude, λ, by G, we have

$$\mathrm{E} : \mathrm{G} :: \frac{b^2}{a} : \frac{a(1-e^2)}{(1-e^2\sin.^2\lambda)^{\frac{1}{2}}}.$$

Hence

$$\mathrm{G}=\mathrm{E}(1+\tfrac{1}{2}e^2.\sin.^2\lambda), \text{ nearly,}$$

or using 2δ for e^2,

$$\mathrm{G}=\mathrm{E}(1+\delta.\sin.^2\lambda).$$

CLAIRAUT *Fig. de la Terre*, p. 191.

PROP. LXIII.

(263.) *To compute that latitude in which the distance from the earth's centre, supposed of small eccentricity, is equal to the radius of the equicapacious sphere.*

The radius, r, of the equicapaeious sphere is $a^{\frac{2}{3}}.b^{\frac{1}{3}}$, since the solid contents of the spheroid is $\frac{4\pi a^2 b}{3}$, and that of a sphere $\frac{4\pi r^3}{3}$. Hence by the polar equation of the ellipse

$$a^{\frac{4}{3}}.b^{\frac{2}{3}}=\frac{b^2}{1-e^2\cos.^2\lambda},$$

or

$$a^{\frac{4}{3}}=b^{\frac{4}{3}}.(1+e^2\cos.^2\lambda), \text{ nearly.}$$

Writing for a, as before, $b.(1+\delta)$, we have

$$1+\tfrac{4}{3}\delta=1+2\delta.\cos.^2\lambda,$$

and consequently

$$\cos.^2\lambda=\tfrac{2}{3}.$$

(264.) At the extremity of this radius, or its production, the attraction is therefore nearly equal to the mass of the earth, divided by the square of the distance of its centre of gravity, and therefore nearly the same as if the earth were a sphere; *see* Art. (238). The polar distance of the parallel on which $\cos.^2\lambda=\frac{2}{3}$ is 54° 44′, English measure. This angle is remarkable for being the result in many physical questions of *maxima* and *minima.*

We shall next show how the masses of the planets, their densities, and the comparative weights of bodies at their surface may be determined. In general there are two ways in which the masses of the planets may be determined; the first, is by the application of Kepler's law relative to periodic times and distances; and the second, by the disturbing forces, as the effects produced by the latter are entirely dependant upon the masses of the disturbing bodies. The former method is applicable only to the planets that have satellites, the latter to the planets in general.

PROP. LXIV.

(265.) *To determine the mass of a planet, accompanied with a satellite, relative to the sun's mass.*

For the planet about the sun we have by the formula, $t=\frac{2\pi a^{\frac{3}{2}}}{\sqrt{M+m}}$. For the satellite about the planet, $t'=\frac{2\pi a'^{\frac{3}{2}}}{\sqrt{m+m'}}$. Hence we have

$$\frac{m+m'}{M+m}=\frac{a'^3}{a^3}\cdot\frac{t^2}{t'^2},$$

or if we reject m' compared with m, and m compared with M, we have

$$\frac{m}{M}=\frac{a'^3}{a^3}\cdot\frac{t^2}{t'^2}.$$

This expression for the ratio of the mass of the planet to that of the sun is perfectly exact, if M, m, and m' be in geometrical progression, which is nearly the case for the sun, Jupiter, and his fourth satellite.

(266.) For certain values of M, m, and m', the above expression might be rendered more accurate by rejecting m' compared with m, but not m compared with M. Hence we have

$$\frac{m}{M}=\frac{a'^3}{a^3}\cdot\frac{t^2}{t'^2}+\left(\frac{a'^3}{a^3}\cdot\frac{t^2}{t'^2}\right)^2, \text{ nearly};$$

or if s denote the sine of the angle under which, at the planet's mean distance from the sun, the mean radius of the satellites orbit is seen, we have

$$\frac{m}{\mathrm{M}}=s^3.\frac{t^2}{t'^2}+\left(s^3.\frac{t^2}{t'^2}\right)^2.$$

(267.) In place of using the period and distance of the planet whose mass is sought, Newton and his commentators use the period and distance of Venus. Let Q and d denote this period and distance, and we have

$$\frac{m}{\mathrm{M}}=\frac{a'^3}{d^3}.\frac{\mathrm{Q}^2}{t'^2},$$

or more accurately,

$$\frac{m}{\mathrm{M}}=\frac{a'^3}{d^3}.\frac{\mathrm{Q}^2}{t'^2}+\left(\frac{a'^3.\mathrm{Q}^2}{d^3.t'^2}\right)^2.$$

The advantage of this method consists in this, that the fraction $\frac{\mathrm{Q}^2}{d^3}$, when once computed, answers as a factor in the expression for the mass of every planet that has a satellite. It is also on some occasions more exact, as the mass of Venus is so small compared with the sun. Woodhouse is not strictly correct in his statement, that this process requires an additional step.

PROP. LXV.

(268.) *Knowing the moon's horizontal parallax*, P, *and periodic time*, t'; *also, the sun's horizontal parallax*, Π, *and length of a sidereal year*, t, *to find the relation of the masses of the earth and sun.*

By Art. (265.) we have $\frac{m}{\mathrm{M}}=\frac{a'^3.t^2}{a^3.t'^2}$, but a' is equal to the sun's distance into the sine of the moon's greatest elongation as seen from the sun, that is, equal to $a.\frac{\Pi}{\mathrm{P}}$, hence

$$\frac{m}{\mathrm{M}}=\frac{t^2}{t'^2}.\left(\frac{\Pi}{\mathrm{P}}\right)^3.$$

All these quantities can be ascertained with sufficient exactness except Π, and as $\frac{m}{\mathrm{M}} \propto \Pi^3$, it appears that any error on the parallax will cause nearly three times as great an error on the value of the earth's mass.

PROP. LXVI.

(269.) *To determine the mass of the earth relative to that of the sun, by a comparison of the spaces they actually descend one towards the other in the same time.*

Let s denote the deflection from the tangent of the earth's orbit in a second, and g the space fallen towards the earth in the same time at the sun's mean distance, then $\frac{m}{\mathrm{M}}=\frac{g}{s-g}$. Now the space fallen through in a second on the parallel for which the sine of the lat. is $\frac{1}{\sqrt{3}}$, is $16\frac{1}{8}$ feet, or in parts of the radius of the earth's orbit $\frac{16\frac{1}{8}}{a}$, hence $g=\frac{16\frac{1}{8}}{a}.\frac{r^2}{a^2}$ (r denoting the earth's radius) $=\frac{16\frac{1}{8}}{r}.\sin.^3\Pi$. Again, the arc ($z$) described in one second by the earth $=\frac{2\pi a}{t.24.(60)^2}$, and consequently $s=\frac{z^2}{2a}=\frac{2\pi^2 a}{(t.24.60.60)^2}$.

By actual computation of these values we find $g=\frac{5{,}9634}{10^{20}}$, and $s=\frac{1982016}{10^{20}}$, taking the mean radius of the earth's orbit to be unity. Hence

$$\frac{m}{\mathrm{M}}=\frac{5{,}9634}{1982010}.$$

This method, like the preceding, is subject to inaccuracy, from the uncertainty attending the determination of the sun's parallax.

PROP. LXVII.

(270.) *To shew how the densities of the planets and weights of bodies at their surfaces may be determined.*

The densities being directly as the masses, and inversely as the magnitudes, it is only necessary, in order to find the relative densities, to multiply the expressions for the masses by $\left(\frac{\mathrm{R}}{r}\right)^3$, R and r denoting the radii of the sun and planet respectively. Multiply the earth's mass in Prop. (65.) by this, and, denoting the respective densities of the planet and sun by δ and 1, we find for the earth

$$\delta=\frac{t^2}{t'^2}\cdot\frac{(\text{tan.}\ \odot\ \text{ap. semid.})^3}{\mathrm{P}^3}.$$

This expression being independant of the sun's parallax, the determination of the densities is less liable to error than that of the masses.

The densities of the planets are found to be less the more remote they are from the sun. According to Kepler the law of their diminution was the inverse square root of the distance.

(271.) Multiplying $\frac{m}{\text{M}}$ by $\frac{\text{R}^2}{r'^2}$, we find the ratio of a weight on the earth's surface to that of the same body on the surface of the sun. This gives us

$$\frac{w}{w'}=\frac{t^2}{t'^2}\cdot\frac{(\text{tan.} \odot \text{ ap. diam.})^2 . \Pi}{\text{P}^3}.$$

This can be found more accurately than $\frac{m}{\text{M}}$, since it varies as Π, and $\frac{m}{\text{M}}$ as Π^3.

As to the second method of determining the masses of the planets, it is unsuitable to a treatise like the present.

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SECTION VII.

GENERAL PROPERTIES OF THE MOTION OF A SYSTEM OF BODIES.—MOTION OF THE CENTRE OF GRAVITY OF TWO BODIES ABOUT A DISTANT BODY.—PROBLEM OF THREE OR MORE BODIES STATED.

PROP. LXVIII.

(272.) *To apply the principle of D'Alembert to the motion of a system of bodies, whatever be the forces acting.*

Let X, Y, Z be the forces acting on the body, *m*, parallel to the axes respectively; then the impressed increment of velocity in the time, dt, is $\mathrm{X}dt$; the effective increment in the same time is $d.\frac{dx}{dt}$. The difference of these is a velocity which does not shew itself in the system, and $m.\left(\mathrm{X}-\frac{d^2x}{dt^2}\right)$ is a motive force which does not shew itself; similarly for m', $m'.\left(\mathrm{X}'-\frac{d^2x'}{dt^2}\right)$ is a motive force which does not shew itself; these motive forces must consequently equilibrate by the connection of the parts of the system. Hence we have the six equations of equilibrium to be fulfilled by these forces

$$\Sigma m.\frac{d^2x}{dt^2}=\Sigma m\mathrm{X}, \quad \Sigma m.\frac{d^2y}{dt^2}=\Sigma m\mathrm{Y}, \quad \Sigma m.\frac{d^2z}{dt^2}=\Sigma m\mathrm{Z},$$

$$\Sigma m.(x\,d^2y-y\,d^2x) = \Sigma m.(x\mathrm{Y}-y\mathrm{X})\,dt^2,$$

$$\Sigma m.(z\,d^2x-x\,d^2z) = \Sigma m.(z\mathrm{X}-x\mathrm{Z})\,dt^2,$$

$$\Sigma m.(y\,d^2z-z\,d^2y) = \Sigma m.(y\mathrm{Z}-z\mathrm{Y})\,dt^2.$$

(273.) If the forces of the system be such that they would equilibrate on the hypothesis of the system being invariable, the second members of all these equations must vanish; for if the system be invariable, they would vanish by hypothesis, and if it were not, since the forces and points of application continue the same, they should continue to vanish.

If *some* only of the forces possess the above character, these may be neglected in forming the second members. Of this number are all forces arising from the mutual attraction of the parts of the system; since, by the law of action and re-action, if the only forces acting were these attractions, the system would remain at rest if its form were invariable.

PROP. LXIX.

(274.) *If there be no fixed point in the system, the centre of gravity moves as if the several bodies were collected therein, and the several forces there applied parallel to their primitive directions.*

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of gravity, then

$$M\bar{x} = \Sigma mx, \qquad M\bar{y} = \Sigma my, \qquad M\bar{z} = \Sigma mz,$$

M denoting the sum of the bodies. Differentiating twice

$$M.\frac{d^2\bar{x}}{dt^2} = \Sigma m.\frac{d^2x}{dt^2}, \quad M.\frac{d^2\bar{y}}{dt^2} = \Sigma m.\frac{d^2y}{dt^2}, \quad M.\frac{d^2\bar{z}}{dt^2} = \Sigma m.\frac{d^2z}{dt^2},$$

whence we have

$$M\frac{d^2\bar{x}}{dt^2} = \Sigma mX, \quad M.\frac{d^2\bar{y}}{dt^2} = \Sigma mY, \quad M.\frac{d^2\bar{z}}{dt^2} = \Sigma mZ.$$

These are the equations of motion of the centre of gravity, the system not containing a fixed point. When there are in the system points which, without being absolutely fixed, are constrained to move on curves or surfaces, we ought to comprise amongst the forces other forces equal and contrary to the pressures which these undergo, and then consider the points absolutely free.

PROP. LXX.

(275.) *To prove that the motion of the centre of gravity is rectilinear and uniform, when the bodies of the system are subjected merely to their mutual actions.*

In this case, by Art. (272.), we have $\Sigma m\text{X}$, $\Sigma m\text{Y}$, $\Sigma m\text{Z}$, each nothing, and consequently $\frac{d^2\bar{x}}{dt^2}$, &c. each cypher. Hence it follows, by Art. (17.), that the motion of the centre of gravity is rectilinear and uniform. Thus, as a material point cannot change its motion without the intervention of some extraneous cause, so a system of bodies cannot alter the motion of its centre of gravity by the mere action of its component parts. This important result constitutes that mechanical principle denominated, *the general principle of the conservation of the motion of the centre of gravity.*

(276.) Conversely it follows, that the relative motions of the parts of a system are the same, whether the system be at rest, or be carried *uniformly* forward with any velocity in any *rectilinear* direction.

(277.) If, moreover, during the motion two or more of the bodies should come together, this percussion will not change the velocity of the centre of gravity, nor the direction of its motion. To prove this, let a, b, c be the components of m's velocity before the stroke, and A, B, C after; then we have in the former state

$$\text{M}.\frac{d\bar{x}}{dt}=\Sigma ma,\quad \text{M}.\frac{d\bar{y}}{dt}=\Sigma mb,\quad \text{M}.\frac{d\bar{z}}{dt}=\Sigma mc,$$

and in the latter,

$$\text{M}.\frac{d\bar{x}}{dt}=\Sigma m\text{A},\quad \text{M}.\frac{d\bar{y}}{dt}=\Sigma m\text{B},\quad \text{M}.\frac{d\bar{z}}{dt}=\Sigma m\text{C}.$$

By the theory of the collision of bodies, equilibrium ought to exist in the system between the quantities of motion lost and gained; this gives us, since the system contains no fixed point, $\Sigma ma - \Sigma mA = 0$, and consequently $\frac{d\bar{x}}{dt}$, the same before and after the stroke; the same is true for $\frac{d\bar{y}}{dt}$ and $\frac{d\bar{z}}{dt}$, and consequently the velocity and direction remain the same.

PROP. LXXI.

(278.) *In a system of bodies subjected merely to their mutual attractions, and not containing any fixed point, the sum that arises by multiplying each body into the area, traced out on any plane about any point in space, is constant in a given time.*

Let $\frac{1}{2}\lambda$ be this sum for the time t on the plane of x, y, taken in any manner through the point in question, then $\Sigma m(x\,dy - y\,dx) = d\lambda$; let also $\frac{1}{2}\lambda'$, $\frac{1}{2}\lambda''$ be the analogous sums on the planes of x, z and y, z, then $\Sigma m(z\,dx - x\,dz) = d\lambda'$, $\Sigma m(y\,dz - z\,dy) = d\lambda''$. From these we obtain, by differentiating a second time,

$$\Sigma . m(x\,d^2y - y\,d^2x) = d^2\lambda,$$

$$\Sigma . m(z\,d^2x - x\,d^2z) = d^2\lambda',$$

$$\Sigma . m(y\,d^2z - z\,d^2y) = d^2\lambda''.$$

Hence, from Art. (272.), we have

$$d^2\lambda = \Sigma m . (xY - yX)\,dt^2,$$

$$d^2\lambda' = \Sigma m . (zX - xZ)\,dt^2,$$

$$d^2\lambda'' = \Sigma m . (yZ - zY)\,dt^2.$$

But since the only forces acting are the mutual attractions, we have the second members of these equations cypher hence, by integrating

$$\lambda=ht, \qquad \lambda'=h't, \qquad \lambda''=h''t.$$

In this consists *the general principle of the conservation of areas.*

(279.) If there be a fixed point in the system, this theorem is not true, unless this point be made the origin of coordinates; but the direction of the planes on which these areas are projected is still arbitrary. If there be two fixed points in the system, and that we take the line joining them for the axis of z, we shall have then $\lambda=ht$, and the principle of areas will not be true unless the origin be taken on this axis, and the plane of projection perpendicular thereto. In the case of a single fixed centre, independant of the mutual attractions, there may be any number of forces directed to this centre, and the theorem of areas would still continue true, for the moments of all such forces are nothing in any plane through the origin.

PROP. LXXII.

(280.) *To determine the position of the maximum or invariable plane in a system of bodies subjected to their mutual attractions, or to forces directed to the origin of coordinates.*

The area described on each of the coordinate planes in any instant is the projection of the area described in space in the same time by one of the bodies, consequently $\frac{1}{2}\lambda$, $\frac{1}{2}\lambda'$, $\frac{1}{2}\lambda''$ are the sums of the projections of the areas described in actual space. Let $\frac{1}{2}L$, $\frac{1}{2}L'$, $\frac{1}{2}L''$ be the same on three other coordinate planes through the same origin, also, at right angles, then we have

$$L=\lambda.\cos.L,\lambda+\lambda'.\cos.L,\lambda'+\lambda''.\cos.L,\lambda'',$$
$$L'=\lambda.\cos.L',\lambda+\lambda'.\cos.L',\lambda'+\lambda''.\cos.L',\lambda'',$$
$$L''=\lambda.\cos.L'',\lambda+\lambda'.\cos.L'',\lambda'+\lambda''.\cos.L'',\lambda''.$$

From these equations we can readily deduce

$$L^2+L'^2+L''^2=\lambda^2+\lambda'^2+\lambda''^2.$$

This shews us that the sum of the squares of the projections is constant on any three rectangular planes through the same origin.

It is obvious that L would be a *maximum* if L' and L'' were each cypher, and this being the case, we should have $L^2=\lambda^2+\lambda'^2+\lambda''^2$; $L.\cos.L,\lambda=\lambda$; $L.\cos.L,\lambda'=\lambda'$; $L.\cos.L,\lambda''=\lambda''$. This is always possible, since the values for the cosines of the angles which determine this plane are always possible, and always proper fractions. Substituting for λ, λ', λ'', their proper values ht, $h't$, $h''t$, we have

$$\cos.L,\lambda=\frac{h}{\sqrt{h^2+h'^2+h''^2}},$$

$$\cos.L,\lambda'=\frac{h'}{\sqrt{h^2+h'^2+h''^2}},$$

$$\cos.L,\lambda''=\frac{h''}{\sqrt{h^2+h'^2+h''^2}}.$$

These angles are therefore constant, which proves that the maximum plane will constantly preserve the same position, as long as the only forces acting are the mutual attractions and forces directed to the centre of areas. It is for this reason that Laplace, to whom the discovery of it is due, has called it the *invariable plane*.

When there is no fixed point in the system, and no force but the mutual attractions, there is a particular invariable plane for each point in space.

PROP. LXXIII.

(281.) *To construct the invariable plane at any instant, knowing the velocities and coordinates of the several bodies.*

The position of this plane depends on the quantities h, h', h''; and the values of these are determinable from the given *data* by the equations of Prop. (71.), since these are the same as $\frac{d\lambda}{dt}, \frac{d\lambda'}{dt}, \frac{d\lambda''}{dt}$.

The quantities h, h', h'' are the sums of the moments on the respective planes, and consequently the invariable plane here determined is the same as the plane of the resultant pair.—*See* Lloyd's *Mech., sect.* iv.

The consideration of the invariable plane is useful in Dynamics, for by taking it for one of the coordinate planes, two of the three arbitrary constants, h, h', h'', will vanish, which will render more easy the treatment of the equations into which these constants enter.

PROP. LXXIV.

(282.) *To compare with each other the principal planes relative to different origins, and to determine when they remain parallel.*

Let H, H′, H″ be what h, h', h'' become relative to the new origin, and let the coordinates of this new origin be α, β, γ, then we have

$$\mathrm{H}=\Sigma m.\left\{(x-\alpha)\frac{dy}{dt}-(y-\beta)\frac{dx}{dt}\right\}$$

$$=\Sigma m.\left(x\frac{dy}{dt}-y.\frac{dx}{dt}\right)-\alpha.\,\Sigma m.\frac{dy}{dt}+\beta.\,\Sigma m.\frac{dx}{dt}$$

$$=h+\mathrm{M}.\left(\beta.\frac{d\bar{x}}{dt}-\alpha.\frac{d\bar{y}}{dt}\right).$$

Also, we have

$$\mathrm{H}'=h'+\mathrm{M}.\left(\alpha.\frac{d\bar{z}}{dt}-\gamma.\frac{d\bar{x}}{dt}\right)$$

$$\mathrm{H}''=h''+\mathrm{M}.\left(\gamma.\frac{d\bar{y}}{dt}-\beta.\frac{d\bar{z}}{dt}\right).$$

(283.) We shall have $\mathrm{H}=h$, $\mathrm{H}'=h'$, $\mathrm{H}''=h''$, and consequently the invariable planes respectively parallel, if the three following equations hold,

$$\beta.\frac{d\bar{x}}{dt}-\alpha.\frac{d\bar{y}}{dt}=0,\quad \alpha.\frac{d\bar{z}}{dt}-\gamma.\frac{d\bar{x}}{dt}=0,\quad \gamma.\frac{d\bar{y}}{dt}-\beta.\frac{d\bar{z}}{dt}=0.$$

In the motion of the system it has been already shewn that the centre of gravity moves uniformly, and in a right line, the components $\frac{d\bar{x}}{dt}$, $\frac{d\bar{y}}{dt}$, $\frac{d\bar{z}}{dt}$ of its velocity are constant, and the above equations will consequently be verified for all points on any line parallel to a line through the centre of gravity. Consequently the invariable planes remain parallel for all points along this line parallel to the line through the centre of gravity. Also, the invariable plane relative to the centre of gravity always preserves the same position as this centre moves forward.

PROP. LXXV.

(284.) *The invariable plane through the centre of gravity may be determined by the relative velocities about this centre, without knowing the absolute velocities in space, and by the coordinates relative to this centre.*

Changing the origin to the centre of gravity we have $\alpha=\bar{x}$, $\beta=\bar{y}$, $\gamma=\bar{z}$, and consequently

$$\mathrm{H}=\Sigma m.\left\{(x-\bar{x})\frac{dy}{dt}-(y-\bar{y})\frac{dx}{dt}\right\};$$

but we have also

$$\Sigma m.(x-\bar{x})\frac{d\bar{y}}{dt}=(\Sigma mx-\mathrm{M}\bar{x})\frac{d\bar{y}}{dt}=0,$$

$$\Sigma m.(y-\bar{y})\frac{d\bar{x}}{dt}=(\Sigma my-\mathrm{M}\bar{y})\frac{d\bar{x}}{dt}=0.$$

Subtracting these cypher values from the value of H, we have

$$\mathrm{H}=\Sigma m.\left\{(x-\bar{x})\left(\frac{dy}{dt}-\frac{d\bar{y}}{dt}\right)-(y-\bar{y})\left(\frac{dx}{dt}-\frac{d\bar{x}}{dt}\right)\right\}.$$

Similar values may be formed for H′ and H″; and consequently we have the direction of the invariable plane depending on the relative velocities $\frac{dx}{dt}-\frac{d\bar{x}}{dt}$, &c. and on the relative coordinates $x-\bar{x}$, &c.

This important Theorem enables us to determine the position of this plane in the planetary system with great facility.

PROP. LXXVI.

(285.) *The position of the invariable plane will remain unaltered if two, or a greater number, of the bodies of the system should mutually impinge.*

On the principle that the quantity of motion lost by the impact is nothing, we must have, retaining the notation of Prop. (.70),

$$\Sigma m.\{x.(b-\mathrm{B})-y(a-\mathrm{A})\}=0,$$

hence we have

$$\Sigma m.(xb-ya)=\Sigma m.(x\mathrm{B}-y\mathrm{A});$$

but the first number of this equation is the value of h before impact, and the second its value after; consequently, h remains unaltered. The same is true of h' and h''.

Thus we see that the principle of the conservation of areas is independant of any changes, however abrupt, of the velocities produced by the mutual action of the bodies of the system.

Some of the results in the preceding articles might have been arrived at by the theory of *pairs* or *couples*. But, in the first place, this theory gives no calculated results, and would only prove mere facts, so that the business would still remain exceedingly incomplete; and secondly, being established and proved, as it is in the Mechanics of POINSOT and LLOYD, for a system of *invariable* form, the transition thence to one of *variable* form is neither easy nor natural.

(286.) The general principles of the conservation of the motion of the centre of gravity and of areas, are no more than an extension of the six equations of motion of a system of invariable form to any system whatsoever; but the principles which we shall next proceed to develope, are by no means comprised in these equations, but rather results of another statical principle, namely, that of virtual velocities.

PROP. LXXVII.

(287.) *In a system of bodies subjected to the action of any forces whatsoever, to deduce the equation which results from combining the principle of D'Alembert with that of virtual velocities.*

Since $Xdt-d.\frac{dx}{dt}$, $Ydt-d.\frac{dy}{dt}$, $Zdt-d.\frac{dz}{dt}$; $X'dt-d.\frac{dx'}{dt}$, &c. are so many velocities that do not shew themselves in the system in consequence of the mutual connection of the parts thereof, it follows that the motive forces depending on these velocities must equilibrate. Hence, applying to these motive forces the principle of virtual velocities, we have

$$\Sigma \left\{ m\left(Xdt-d.\frac{dx}{dt}\right)\delta x+m.\left(Ydt-d.\frac{dy}{dt}\right)\delta y+ m.\left(Zdt-d.\frac{dz}{dt}\right)\delta z \right\} = 0.$$

In which δx, δy, δz denote the virtual velocities, or possible simultaneous displacements of the body, m, along the axes respectively.

(288.) This equation holds when the bodies undergo an abrupt change in their velocities, that is, when the velocities lost are finite quantities; as well as when the law of continuity is not interrupted, or the velocities lost infinitely small. It holds for all the displacements the system can possibly undergo, without violating the conditions which connect the parts one with another, and which may restrict some of the bodies to move on given surfaces or curves. In each case this equation will give as many particular equations as there are possible motions; and the aggregate of these, with the equations which it may be necessary for δx, $\delta x'$, &c. δy, $\delta y'$, &c. δz, $\delta z'$, &c. to fulfil, will serve to determine the motion of the system.

PROP. LXXVIII.

(289.) *The general principle of the conservation of the vis viva deduced from the above general equation.*

The coordinates of m at any instant being x, y, z, those of m', at the same instant, x', y', z', &c., we may suppose the conditions of the system expressed by equations between these coordinates. These equations may contain the variable t, this would be the case, for instance, if one of the bodies was constrained to a surface which is itself in motion according to a given law. Let then $F(t, x, y, \&c.)$ be one of these equations. Now, in the first place, in order that the arbitrary displacements of the bodies of the system should be lawful this equation must hold, writing $x+\delta x$, $y+\delta y$, $z+\delta z$, &c. for the variables, and consequently its differential, considering t constant, must equal cypher. Hence we have

$$\frac{dF}{dx}\cdot\delta x+\frac{dF}{dy}\cdot\delta y+\frac{dF}{dz}\cdot dz+\frac{dF}{dx'}\cdot\delta x'+\&c.=0.$$

But the coordinates of the several bodies are also functions of t, and as such must at each instant fulfil the above equation. Hence, considering t as variable, we have

$$\mathrm{T}dt+\frac{dF}{dx}\cdot dx+\frac{dF}{dy}\cdot dy+\&c.=0.$$

Where T denotes the differential coefficient of the function F relative to t. Comparing this equation with the preceding, they will agree perfectly if $\mathrm{T}dt=0$, and that we take $dx=\delta x$, $dy=\delta y$, &c.

In this case, namely, when no one of the equations of condition of the system contains the time explicitly, we may suppose the virtual velocities along the axes of coordinates equal to the differentials of these coordinates, which are the spaces passed over by the projections of the several bodies on the axes respectively. This hypothesis means that we may conceive impressed on the system the motions which its parts really have, which could not be if any part had any additional cause of motion, such as, for instance, being restricted to a line or surface which was itself in motion from some extraneous cause. In such a case one of the equations of condition should involve the time explicitly.

When the time is not thus involved, the equation of Art. (287.) may be written

$$\Sigma m.\left(\frac{dx.d^2x+dy.d^2y+dz.d^2z}{dt^2}\right)=\Sigma m.(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz).$$

If v be the velocity of m we have $v^2.dt^2=dx^2+dy^2+dz^2$, hence, integrating the preceding equation, we have, if the second member be integrable,

$$\Sigma mv^2=\beta+2.\phi(x, y, z, x', y', z', \&c.)$$

Thus we see how the theorem of Art. (38.) has been extended to a system of bodies. This theorem thus extended is called, *the general principle of the conservation of vires vivæ.*

PROP. LXXIX.

(290.) *To determine in what cases the formula* $\Sigma m.(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$ *is integrable.*

This formula is integrable, as has been shewn in Art. (41.), when the forces are directed to any number of fixed centres, and functions of their distances from these centres.

This theorem may also be proved as follows:—let the whole force which acts on m, in consequence of the action of such centres, have for its components X, Y, Z; then the amount of this whole force, s being a part of the line along which it acts, will be

$$\mathrm{X}.\frac{dx}{ds}+\mathrm{Y}.\frac{dy}{ds}+\mathrm{Z}.\frac{dz}{ds}.$$

Again, let F, F′, &c. be the forces directed to the fixed centres, and their resultant is

$$\mathrm{F}.\frac{df}{ds}+\mathrm{F}'.\frac{df'}{ds}+\&c.$$

consequently

$$Xdx+Ydy+Zdz=Fdf+F'df'+\&c.$$

The same being applied to the other bodies, we have generally

$$\Sigma m.(Xdz+Ydy+Zdz)=\Sigma m.(Fdf+F'df'+\&c.)$$

See **Prony**, *Mec. Anal.*, Part II. p. 117. It is also integrable when the acting forces are the mutual attractions of the bodies of the system, and that these attractions may be expressed in functions of the masses and distances. Let, in fact, δ be the distance of the bodies m, m'; Δ a function of δ; $m'\Delta$ the accelerating force on m, arising from the action of m'; $m\Delta$ the same on m' from the action of m. Considering only the components of the force on m, we have

$$Xdx+Ydy+Zdz=\frac{m'\Delta}{\delta}\cdot\{(x'-x)dx+(y'-y)dy+(z'-z)dz\}.$$

Similarly for the components on m',

$$X'dx'+Y'dy'+Z'dz'=\frac{m\Delta}{\delta}\cdot\{(x-x')dx'+(y-y')dy'+(z-z')dz'\}.$$

Multiplying the first of these quantities by m, and the second by m', and adding them, we find that the mutual attraction of m and m' introduces into the formula $\Sigma m(Xdx+Ydy+Zdz)$, the term

$$\frac{mm'\Delta}{\delta}\cdot\{(x'-x).(dx-dx')+(y'-y).(dy-dy')+(z'-z).(dz-dz)\};$$

but

$$\delta^2=(x-x')^2+(y-y')^2+(z-z')^2,$$

whence this term becomes, by differentiating δ^2,

$$-mm'\Delta d\delta,$$

a quantity which is a function of δ, and consequently integrable.

Thus the equation of Art. (289.) holds when the bodies of the system are subjected to their mutual attractions, or to the attractions of fixed centres, however these bodies are otherwise restricted; if, for instance, some or all of them be restricted to move on given curves or surfaces. In such a system the sum of the *vires vivæ*, Σmv^2, will be given for any one set of values of x, y, z when we know it for any other. The change on this sum will be independant of the curves described by the several bodies, and this sum will always become the same when the bodies of the system return to the same position; and this sum will always remain constant when the moving bodies are not solicited by any accelerating force.

PROP. LXXX.

(291.) *To estimate the loss of vis viva, occasioned by the collision of several bodies of the system.*

Let a, b, c be the components parallel to the axes of the velocity of m before the collision; A, B, C the same after. The components of the velocity lost are then $a-A$, $b-B$, $c-C$; and consequently at the moment of the abrupt change

$$Xdt-\frac{d^2x}{dt}=a-A,\quad Ydt-\frac{d^2y}{dt}=b-B,\quad Zdt-\frac{d^2z}{dt}=c-C.$$

Immediately after the blow, the projections of m on the axes describe the spaces Adt, Bdt, Cdt during the instant dt; consequently the values of dx, dy, dz which enter into the equation of Art. (287). This equation then becomes

$$\Sigma m\{(a-A)A+(b-B)B+(c-C)C\}=0.$$

If we write v^2 for $a^2+b^2+c^2$, V^2 for $A^2+B^2+C^2$, and u^2 for $(a-A)^2+(b-B)^2+(c-C)^2$, this equation may readily be reduced to the form

$$\Sigma m v^2 - \Sigma m V^2 = \Sigma m u^2.$$

This shews us that the loss of *vis viva* caused by the shock is equal to the *vis viva* due to the loss of velocity.

PROP. LXXXI.

(292.) *In the motion of a system of bodies for which the principle of vis viva holds true, if we take the sum of the products that arise by multiplying each body into its velocity into the element of the path it describes, and if we integrate this sum between two given positions of the system, the value of this integral is a minimum.*

The thing to be proved is, that the integral of $\Sigma m v ds$ taken between given limits is a *minimum*, or that its variation is cypher. We have $\delta\int\Sigma m v ds = \int\Sigma m\,\delta.\,v ds$ by the principles of the calculus of variations; and $\Sigma m\,\delta.\,v ds = \Sigma m v.\,\delta ds + \Sigma m.\,\delta v.\,ds$; but we have also $ds = v dt$, and therefore

$$\delta v.\,ds = v.\,\delta v.\,dt = \frac{dt}{2}\cdot\delta.\,v^2, \text{ and } \Sigma m.\,\delta v.\,v ds = \frac{dt}{2}\cdot\Sigma m.\,\delta.\,v^2.$$

By Art. (289.) we have

$$\Sigma m.\,\delta.\,v^2 = 2\,\Sigma m.\,(X\delta x + Y\delta y + Z\delta z)\cdot$$

Moreover, by the equation of Art. (287.) we have

$$dt.\Sigma m.(X\delta x+Y\delta y+Z\delta z)=\Sigma m.\left(\frac{d^2x}{dt}\cdot\delta x+\frac{d^2y}{dt}\cdot\delta y+\frac{d^2z}{dt}\cdot\delta z\right)$$

whence we have

$$\Sigma m.\delta v.ds=\Sigma m.\left(\frac{d^2x}{dt}\cdot\delta x+\frac{d^2y}{dt}\cdot\delta y+\frac{d^2z}{dt}\cdot\delta z\right), \qquad (a)$$

but we have also*

$$v.\delta.ds=\frac{dx}{dt}\cdot d.\delta x+\frac{dy}{dt}\cdot d.\delta y+\frac{dz}{dt}\cdot d.\delta z,$$

and consequently

$$\Sigma mv.\delta.ds=\Sigma m.\left(\frac{dx}{dt}\cdot d.\delta x+\frac{dy}{dt}\cdot d.\delta y+\frac{dz}{dt}\cdot d.\delta z\right), \qquad (b)$$

adding together equations (*a*) and (*b*) we find

$$\Sigma m.\delta.vds=\Sigma m.d.\left(\frac{dx}{dt}\cdot\delta x+\frac{dy}{dt}\cdot\delta y+\frac{dz}{dt}\cdot\delta z\right).$$

Integrating this we find

$$\delta\int\Sigma mvds=\int\Sigma m\delta.vds=\Sigma m.\left(\frac{dx}{dt}\cdot\delta x+\frac{dy}{dt}\cdot\delta y+\frac{dz}{dt}\cdot\delta z\right).$$

The right hand member of this equation is nothing for the two limits of the integral $\int\Sigma mvds$; for at these limits

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* It may be proved thus:—$ds^2=dx^2+dy^2+dz^2$, hence,

$$\delta.ds=\frac{ds}{dx}.d.\delta x+\frac{dy}{ds}.d.\delta y+\frac{dz}{ds}.d.\delta z,$$

and since $ds=vdt$, we have

$$v.\delta.ds=\frac{dx}{dt}.d.\delta x+\frac{dy}{dt}.d.\delta y+\frac{dz}{dt}.d.\delta z.$$

the positions of all the points of the system being given, the variations of their coordinates ought to vanish, so that the values of δx, δy, δz ought to be nothing for these points.

The general principle established in this article is denominated *the principle of least action.*

(293.) The integral $\int \Sigma mvds$ is the same thing as $\Sigma \int mv^2dt$; but $\Sigma . \int mv^2dt$ is the sum of the *vires vivæ* of the bodies of the system during the continuance of the motion; the principle of least action consists then in saying that the sum of the *vires vivæ* of the system, during the time of passing from one position to another, is a *minimum.* Fermat was mistaken with respect to the principle of least action; his notion of the economy of nature was, that every system of bodies passes from one given position to another in shorter *time* than if the component bodies were to go by any other paths. He reconciled this fact with the law of refraction of light, by supposing the velocity in the medium less than *in vacuo.* Maupertuis, guided by Newton's discoveries on light, found this hypothesis to be erroneous, and that the economy of nature really consisted in giving the sum of the spaces into the velocities a *minimum* for the same body. Euler extended this theorem to variable motions, and it was afterwards generalised and deduced from the laws of motion by Lagrange. The two principles of *vis viva* and of least action differ from the two principles of the conservation of motion of centre of gravity and areas in this, that the two latter are independant of all abrupt changes in the system, but not so the two former.

PROP. LXXXII.

(294.) *To deduce the equations of motion of a system of bodies, the law of force being, as in nature, the inverse square of the distance.*

Let m, m', m'', &c. be the bodies of the system, considered as so many material points. Let x, y, z be the coordinates of m, as before. Then the action of m' on m will be by hypothesis

$$\frac{m'}{(x'-x)^2+(y'-y)^2+(z'-z)^2}.$$

The component of this force parallel to the axis of x will be

$$\frac{m'(x'-x)}{\{(x'-x)^2+(y'-y)^2+(z'-z)^2\}^{\frac{3}{2}}}.$$

This may be written

$$\frac{1}{m}\cdot\left\{d.\frac{m\,m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}}\times\frac{1}{dx}\right\}.$$

Similarly the action of m'' on m, parallel to the axis of x, is

$$\frac{1}{m}\cdot\left\{d.\frac{m\,m''}{\sqrt{(x''-x)+(y''-y)^2+(z''-z)^2}}\times\frac{1}{dx}\right\}.$$

And consequently, if we assume

$$\lambda=\Sigma.\frac{m\,m'}{\sqrt{(x'-x)^2+(y'-y)^2+(z'-z)^2}},$$

we shall have for the whole accelerating force on m parallel to the axis of x,

$$\frac{1}{m}\cdot\frac{d\lambda}{dx}.$$

The same reasoning applies to the forces parallel to the other axes, whence, by the differential equations of motion, we have

$$m.\frac{d^2x}{dt^2}-\frac{d\lambda}{dx}=0.$$

$$m.\frac{d^2y}{dt^2}-\frac{d\lambda}{dy}=0.$$

$$m.\frac{d^2z}{dt^2}-\frac{d\lambda}{dz}=0:$$

Similar equations hold true for the other bodies of the system.

(295.) The determination of the motions of the bodies m, m', &c. depends on the integration of these equations. It has not been found possible to integrate them completely, unless in some few cases, as when the system is composed of two bodies. Some few exact integrals may be deduced, as has been shewn in the preceding Props., without any regard to the law of force. These we might again deduce in the present particular instance by a method somewhat different, as one instance will serve to shew.

PROP. LXXXIII.

(296.) *To determine the motion of the centre of gravity of a system of bodies, the force of mutual attraction being inversely as the square of the distance.*

By the nature of the function λ, we have

$$\frac{d\lambda}{dx}+\frac{d\lambda}{dx'}+\frac{d\lambda}{dx''}+\&c.=0.$$

Hence,

$$\Sigma m.\frac{d^2x}{dt^2}=0.$$

Similarly,

$$\Sigma m.\frac{d^2y}{dt^2}=0, \qquad \Sigma m.\frac{d^2z}{dt^2}=0.$$

Let $\bar{x}$, $\bar{y}$, $\bar{z}$ be the coordinates of the centre of gravity, then

$$\bar{x}=\frac{\Sigma mx}{\Sigma m},$$

consequently,

$$\frac{d^2\bar{x}}{dt^2}=\frac{\Sigma m.d^2x}{(\Sigma m).dt^2}=0.$$

For the same reason $\frac{d^2\bar{y}}{dt^2}=0$, $\frac{d^2\bar{z}}{dt^2}=0$, therefore, &c.

The others may be arrived at in a similar manner, but as we have already obtained them by a general method, it is unnecessary to dwell more on the particular case.

(297.) The preceding are the only exact integrals that analysts have been able to arrive at relative to the motions of a system. Approximation is the only resource in this case, and the constitution of the world affords some facilities for its application. One of the chief is the distribution of the solar system into smaller systems, consisting each of a planet and its satellites, and the great distance of the sun from the primary of each system compared with the mutual distance of the bodies of this system. We shall enter upon this matter to a small extent.

quently its differential, considering t constant, must equal cypher. Hence we have

$$\frac{dF}{dx}\cdot\delta x+\frac{dF}{dy}\cdot\delta y+\frac{dF}{dz}\cdot dz+\frac{dF}{dx'}\cdot\delta x'+\&c.=0.$$

But the coordinates of the several bodies are also functions of t, and as such must at each instant fulfil the above equation. Hence, considering t as variable, we have

$$\mathrm{T}dt+\frac{dF}{dx}\cdot dx+\frac{dF}{dy}\cdot dy+\&c.=0.$$

Where T denotes the differential coefficient of the function F relative to t. Comparing this equation with the preceding, they will agree perfectly if $\mathrm{T}dt=0$, and that we take $dx=\delta x$, $dy=\delta y$, &c.

In this case, namely, when no one of the equations of condition of the system contains the time explicitly, we may suppose the virtual velocities along the axes of coordinates equal to the differentials of these coordinates, which are the spaces passed over by the projections of the several bodies on the axes respectively. This hypothesis means that we may conceive impressed on the system the motions which its parts really have, which could not be if any part had any additional cause of motion, such as, for instance, being restricted to a line or surface which was itself in motion from some extraneous cause. In such a case one of the equations of condition should involve the time explicitly.

When the time is not thus involved, the equation of Art. (287.) may be written

$$\Sigma m.\left(\frac{dx.d^2x+dy.d^2y+dz.d^2z}{dt^2}\right)=\Sigma m.(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz).$$

If v be the velocity of m we have $v^2.dt^2=dx^2+dy^2+dz^2$, hence, integrating the preceding equation, we have, if the second member be integrable,

$$\Sigma mv^2=\beta+2.\phi(x, y, z, x', y', z', \&c.)$$

Thus we see how the theorem of Art. (38.) has been extended to a system of bodies. This theorem thus extended is called, *the general principle of the conservation of vires vivæ.*

PROP. LXXIX.

(290.) *To determine in what cases the formula* $\Sigma m.(\mathrm{X}dx+\mathrm{Y}dy+\mathrm{Z}dz)$ *is integrable.*

This formula is integrable, as has been shewn in Art. (41.), when the forces are directed to any number of fixed centres, and functions of their distances from these centres.

This theorem may also be proved as follows:—let the whole force which acts on m, in consequence of the action of such centres, have for its components X, Y, Z; then the amount of this whole force, s being a part of the line along which it acts, will be

$$\mathrm{X}.\frac{dx}{ds}+\mathrm{Y}.\frac{dy}{ds}+\mathrm{Z}.\frac{dz}{ds}.$$

Again, let F, F′, &c. be the forces directed to the fixed centres, and their resultant is

$$\mathrm{F}.\frac{df}{ds}+\mathrm{F}'.\frac{df'}{ds}+\&c.$$

consequently

PROP. LXXXIV.

(298.) *To determine nearly the motions of the centre of gravity of a system of bodies about a very distant body; for instance, of a planet and its satellites about the sun.*

The origin of coordinates being taken at the centre of the very distant body M; let the coordinates of the centre, whose motion is required, be $\bar{x}, \bar{y}, \bar{z}$; let x, y, z; x', y', z', &c. be the coordinates of the bodies m, m', &c. of the system; and let α, β, γ; α', β', γ', &c. be the coordinates of the same bodies relative to the centre of gravity. It has been already shewn, Art. (274.), that the motion of the centre of gravity of a system is the same as if the several bodies of the system were collected therein, and the several forces there applied parallel to their former directions; it has has also been shewn, Art. (275.), that the mutual action of the bodies does not derange the motion of the centre of gravity, nor their mutual attractions alter this motion. Hence, in determining this motion, it is sufficient to consider the mutual action of M, and the bodies m, m', &c.

The action of M on m parallel to the axis of x, and directed from the origin, is $-\frac{M.x}{r^3}$, where $r=\sqrt{x^2+y^2+z^2}$; hence, by Art. (274.), the entire force on the centre of gravity parallel to the axis of x is

$$-\frac{M.\Sigma\frac{mx}{r^3}}{\Sigma m};$$

but

$$\frac{x}{r^3}=\frac{\bar{x}+\alpha}{\{(\bar{x}+\alpha)^2+(\bar{y}+\beta)^2+(\bar{z}+\gamma)^2\}^{\frac{3}{2}}}.$$

Neglecting all powers and products of the second order, and upwards, of the variables α, β, γ; α', &c. and denoting by R the $\sqrt{\bar{x}^2+\bar{y}^2+\bar{z}^2}$, we obtain

$$\frac{x}{r^3}=\frac{\bar{x}}{\mathrm{R}^3}+\frac{\alpha}{\mathrm{R}^3}-3\bar{x}.\frac{(\bar{x}\alpha+\bar{y}\beta+\bar{z}\gamma)}{\mathrm{R}^5}.$$

Hence we have

$$-\frac{\mathrm{M}.\Sigma\frac{mx}{r^3}}{\Sigma m}=-\frac{\mathrm{M}\bar{x}.\Sigma m}{\mathrm{R}^3.\Sigma m}-\frac{\Sigma m\alpha}{\mathrm{R}^3.\Sigma m}+$$

$$3\bar{x}\frac{(\bar{x}.\Sigma m\alpha+\bar{y}.\Sigma m\beta+\bar{z}.\Sigma m\gamma)}{\mathrm{R}^5.\Sigma m};$$

but we have $\Sigma m\alpha$, $\Sigma m\beta$, $\Sigma m\gamma$, each equal to cypher, consequently

$$-\frac{\mathrm{M}.\Sigma\frac{mx}{r^3}}{\Sigma m}=-\frac{\mathrm{M}.\bar{x}}{\mathrm{R}^3}.$$

Similarly the components of the force of M parallel to the other axes are

$$-\frac{\mathrm{M}.\bar{y}}{\mathrm{R}^3},\quad-\frac{\mathrm{M}.\bar{z}}{\mathrm{R}^3};$$

consequently the force of M on the centre of gravity is nearly the same as if all the bodies of the system were collected therein. But as we are estimating the relative motion of the centre of gravity of the system about M, we should conceive applied to each body in an opposite direction that force with which it attracts. This force resulting from the action of m, m', m'', &c. on M, resolved parallel to x, is $\Sigma\frac{mx}{r^3}$; and, neglecting quantities of the second and higher orders, this becomes $\frac{\bar{x}.\Sigma m}{R^3}$. Similarly the components parallel to the other axes are $\frac{\bar{y}.\Sigma m}{R^3}$, $\frac{\bar{z}.\Sigma m}{R^3}$. Hence, by transferring to the centre of gravity these forces with a contrary sign, we shall have for the entire forces soliciting this point,

$$-(M+\Sigma m).\frac{\bar{x}}{R^3}, \quad -(M+\Sigma m).\frac{\bar{y}}{R^3}, \quad -(M+\Sigma m).\frac{\bar{z}}{R^3},$$

which are precisely the same as if Σm were a single body placed at the centre of gravity, and consequently this centre moves as if all the bodies, m, m', &c., were collected therein.

This theorem will appear in a system of three bodies more readily from the following Prop.

PROP. LXXXV.

(299.) *To determine the force acting on the centre of gravity of two bodies, m and m', directed to a very distant body,* M.

Let p and g be the distances of m and m' from their centre of gravity, and π the angle which the line $p+g$ makes with R. Then R being very great compared with p and g, the distance of M from m is $R-p.\cos.\pi$; and from m', $R+g.\cos.\pi$; considering the attraction of M on m as the resultant of two forces, one directed to m', and the other to M, the latter component has for its value $\frac{MR}{(R-p.\cos.\pi)^3}$; the similar component for m' is

$\frac{MR}{(R+g.\cos.\pi)^3}$. Multiplying these by m and m' respectively, and adding them, we obtain the whole motive force on the centre of gravity, and then dividing by $m+m'$, the accelerating force. Expanding them after this by the binomial, we obtain

$$\frac{M}{R^2}+\frac{3M.\cos.\pi}{R^3(m+m')}.(mp-m'g)+\frac{6M.\cos.^2\pi}{R^4(m+m')}.(mp^2+m'g^2)+\&c.$$

Similarly, by the action of M on m, m', we have, acting on the centre of gravity, another force,

$$\frac{m+m'}{R^2}+\frac{3\cos.\pi}{R^3}.(mp-m'g)+\frac{6.\cos.^2\pi}{R^4}.(mp^2+m'g^2)+\&c.$$

The second term in each of these series vanishes by the nature of the centre of gravity, and consequently the whole force urging this point about M, supposed at rest, is

$$\frac{M+m+m'}{R^2}+\frac{6.\cos.^2\pi}{R^4}.\left(\frac{M}{m+m'}+1\right).(mp^2+m'g^2)+\&c.$$

(300.) The first term of this series gives the law for elliptical motion, and the remaining terms give the value of the force which causes a deviation from this law.

PROP. LXXXVI.

(301.) *To compute the force of* m *towards* M, *when* m *is disturbed by the action of* m'.

The whole force with which m is urged towards M is made up of their mutual attractions, and the quantity by which they are drawn together by the decomposed influences of m' on each. For brevity sake let us denote $p+g$ by r; $R-p.\cos.\pi$ by δ; and $R+g.\cos.\pi$ by y. The attraction of m' on M is $\frac{m'}{y^2}$, this we decompose along r and δ; the latter component is $\frac{m'.\delta}{y^3}$, which is part of the force of m to M; the former component is $\frac{m'r}{y^3}$, and the attraction of m' on m is $\frac{m'}{r^2}$, consequently along the line r there acts a force $\frac{m'}{r^2}-\frac{m'r}{y^3}$; let θ be the inclination of r to δ, and for the whole force, F, with which M attracts m, we have

$$F=\frac{M+m}{\delta^2}+\frac{m'\delta}{y^3}+\left(\frac{m'}{r^2}-\frac{m'r}{y^3}\right).\cos.\theta. \qquad (a)$$

(302.) Also, for the force T perpendicular to δ we have

$$T=\left(\frac{m'}{r^2}-\frac{m'r}{y^3}\right).\sin.\theta. \qquad (b)$$

The expressions for these forces given in the second volume of Harte's Laplace (see note, p. 49,) are erroneous, and some false inferences drawn from them. Notwithstanding these, and some few other blemishes, this very learned and elaborate work possesses high claims indeed on the attention of scientific readers.

(303.) Approximate values for F and T may be arrived at as follows. Since $y^2=\delta^2+r^2-2r\delta.\cos.\theta$, and since δ is very great compared with r, we have

$$\frac{1}{y^3}=\frac{1}{\delta^3}.\left\{1-\tfrac{3}{2}.\left(\frac{r^2}{\delta^2}-\frac{2r}{\delta}.\cos.\theta\right)+\&c.\right\}.$$

Substituting this value in equations (a) and (b), we obtain

$$F=\frac{M+m+m'}{\delta^2}+\frac{m'}{r^2}\cdot\cos.\theta-\frac{3m'r^2}{\delta^4}+\frac{2m'r}{\delta^3}\cdot\cos.\theta.$$

$$T=\left(\frac{m'}{r^2}-\frac{m'r}{\delta^3}-\frac{3m'r^2}{\delta^4}\right).\sin.\theta.$$

Values still less approximate are sometimes used, by rejecting all terms of the order $\frac{r}{\delta^3}$, and upwards. The force then which interferes with elliptical motion in the value of F will be $\frac{m'}{r^2}\cdot\cos.\theta$, which is greater than the disturbing force on the centre of gravity in Art. (299). Thus, as far as this force is concerned, the centre of gravity of *m* and *m'* describes more nearly an ellipse about M, than *m* does.

(304.) If we were similarly to compute the attraction of *m'* to *m*, we would find

$$F=\frac{m+m'}{\delta^2}-\frac{Mr}{2\delta^3}-\frac{3Mr}{2\delta^3}\cdot\cos.2\omega.$$

$$T=\frac{3Mr}{2\delta^3}\cdot\sin.2\omega.$$

(305.) This latter case applies to the moon disturbed by the action of the sun in its motion about the earth; M, *m*, *m'* denoting the sun, earth, and moon respectively. The centripetal force of the moon is diminished therefore in syzygy by the quantity $\frac{2M.r}{\delta^3}$, and increased in quadrature by $\frac{2Mr}{2\delta^3}$, half the former quantity; also, since $\frac{M}{\delta^3}=\frac{m}{r^3}\times\left(\frac{☽\,\text{per}}{⊙\,\text{per}}\right)^2$, $\frac{Mr}{\delta^3}=\frac{m}{r^2}.005595$, or $=\frac{M}{r^2}\cdot\frac{1}{179}$, nearly, we have the mean value of the *addititious* force, as it is called, equal to $\frac{1}{179}$th of the moon's terrestrial gravity; at octants the value of the disturbing force along the radius vector is $-\frac{M.r}{2\delta^3}$, and along the tangent $\pm\frac{3Mr}{2\delta^3}$, which is the *max.* value of the tangential force; the disturbing force along the radius vector is cypher when cos. $2\omega=-\frac{1}{3}$ and $\omega=54°44'$; some of the peculiarities of this angle have been already noticed. To enter more deeply into the theory of perturbations would be to transgress the limits assigned to this work; if the reader wishes for further information, he is referred to the very elegant Tract by Professor Airy, of Cambridge, on this subject.

APPENDIX.

SECTION IX.

ON THE MOTION OF BODIES IN MOVEABLE ORBITS, AND ON THE MOTION OF THE APSIDES.

PROP. XLIII.

To cause a body to move in any trajectory revolving about the centre of force, after the same manner as another body in the same trajectory at rest.

LET the body (Fig. 50.) P revolve in the orbit VPK, given in position proceeding from V towards K. From the centre C, let there be drawn continually C*p* equal to CP, and making the angle VC*p* proportional to the angle VCP; and the area which the line C*p* describes will be to that described in the same time by the line CP as the velocity of the describing line C*p* to the velocity of the describing line CP; that is, as the angle VC*p* to the angle VCP, and therefore in a given ratio, and consequently proportional to the time. Since then the area which the line C*p* describes in an immoveable plane is proportional to the time, it is obvious that a body acted on by a centripetal force of a just quantity may revolve with the point *p* in that curve line which that same point *p* describes in the immoveable plane after the manner just stated. Let the angle V*Cu* be made equal to the angle P*Cp*, the line C*u* to the line CV, and the figure *u*C*p* to the figure VCP, and the body being always in *p* will move in the perimeter of the revolving figure *u*C*p*, and will in the same time describe an arc thereof, *up*, as the other body, P, can describe an arc thereunto similar and equal in the quiescent orbit, VPK. Let there be sought then by Cor. 5. of Prop. 5., the centripetal force with which a body could revolve in that curve which the point *p* describes in the immoveable plane, and the problem shall be solved.

PROP. XLIV.

The difference of the forces required to cause two bodies to revolve equally, one in a quiescent orbit, and the other in the same orbit revolving, is in the triplicate ratio of the common altitude inversely.

Let *up*, *pk*, parts of the revolving orbit, be similar and equal to the parts VP, PK of the quiescent orbit; and let the distance of P from K be understood to be evanescent. From the point, *k*, on the right line, *p*C, let fall a perpendicular, and produce it to *m*, making *mr* to *kr* as the angle VC*p* to the angle VCP. Since the altitudes PC, *p*C and KC, *k*C are constantly equal, it is obvious that the increments or decrements of the lines PC and *p*C must be constantly equal; and that consequently if each of the motions of the bodies at P and *p* be decomposed into two, one paracentric, and the other perpendicular to the lines PC, *p*C; the paracentric motions will be equal, and the transverse motions of *p* and P will be as the angular motions of the lines *p*C, PC, or as the angle VC*p* to the angle VCP. In the same time then as the body P, by its compound motion, arrives at K, the body *p*, by an equal paracentric motion, will be moved equally from *p* towards C, and therefore, at the end of that time will be found somewhere in the line *mkr*, a perpendicular through *k* on the line *p*C; and by its transverse motion will acquire a distance from the line *p*C, which is to the distance which the other body, P, acquires from the line PC, as is the transverse motion of the body *p* to the transverse motion of the other body, P. Wherefore, since *kr* is equal to the distance which the body P acquires from the line PC, and since *mr* is to *kr* as the angle VC*p* to the angle VCP, that is, in the ratio of the transverse motions of the bodies *p* and P, it is obvious that at the end of the time the body must be found in *m*. Such would be the case if the bodies *p* and P were to move equally along the lines *p*C and PC, and therefore, to be urged, with equal forces in these directions. But let the angle *p*C*n* be taken, having to *p*C*k* the same ratio as the angle VC*p* to VCP, and let *n*C be equal to *k*C, and the body *p*, at the end of the time, will really be found in *n*; and it is therefore urged with a greater force than the body P, if the angle *n*C*p* be greater than the angle *k*C*p*, that is, if the orbit *upk* either move *in consequentia*, or move *in antecedentia* with a velocity more than double that of CP *in consequentia*; and with a less force if the orbit move slower than this *in antecedentia*. And the difference of the forces is as the interval, *mn*, through which the body, *p*, ought to be transferred by its action in that given space of time. Conceive a circle with the centre, C, and the interval, C*n* or C*k*, cutting the right lines, *mr*, *mn*, produced in *s* and *t*, and the rectangle $mn \times mt$ will be equal to the rectangle $mk \times ms$, and consequently *mn* equal to $\frac{mk \times ms}{mt}$. But since the triangles *p*C*k*, *p*C*n*, the time being given, are given in magnitude, the lines *mr*, *kr*, their difference *mk*, and their sum *ms*, are inversely as the altitude *p*C, and consequently the rectangle $mk \times ms$ is reciprocally as the square of the altitude pC. Also, mt is directly as $\frac{1}{2}mt$, that is, as the altitude pC. These are the *prime* ratios of these lines in their nascent state; and hence $\frac{mk \times ms}{mt}$, or the nascent line mn, and the difference of the forces, which is proportional thereto, is reciprocally as the cube of the altitude pC.

COR. 1. Hence, the difference of the forces in the places P and p, or K and k, is to the force with which a body can revolve in a circle from R to K in the time of P's describing the arc PK in the immoveable orbit, as the nascent line mn to the versed sine of the nascent arc PK, that is, as $\frac{mk \times ms}{mt}$ to $\frac{rk^2}{2kC}$, or as $mk \times ms$ to the square of rk, or as $G^2 - F^2$ to F^2, if G and F be given quantities, having to each other the ratio of the angles VCp, VCP. And consequently, if with the centre C and any interval, CP or Cp, there be described a circular sector equal to the entire area VPC, which the body P has in any time described about the centre of the immoveable orbit; the difference of the forces with which P and p revolve, one in the immoveable, and the other in the moveable orbit, will be to the centripetal force, urged by which a body can uniformly describe said sector in the same time as the area VPC is described, as $G^2 - F^2$ to F^2. For that sector and the area pCk are to each other as the times in which they are described.

COR. 2. If the orbit VPK be an ellipse, having its focus at C, and its higher apside at V, and an ellipse, upk, be constructed equal and similar thereto, so that pC be always equal to PC, and the angle VCp to the angle VCP in a given ratio, as G to F; if for the altitude PC or pC there be written A, and 2R for the latus rectum of the ellipse, the force with which the body can revolve in the moveable ellipse will be as $\frac{F^2}{A^2} + \frac{RG^2 - RF^2}{A^3}$, and conversely. For let $\frac{F^2}{A^2}$ expound the force with which the body could revolve in the unmoved ellipse, and at V the force will be $\frac{F^2}{CV^2}$. But the force in a circle with the same velocity at the distance CV, is to the force in the ellipse at the apside V, as the semi-*latus rectum* of the ellipse to the semi-diameter of the circle, and it is therefore equal to $\frac{RF^2}{VC^3}$, and the force, which is to this as G^2-F^2 to F^2 is equal to $\frac{RG^2-RF^2}{CV^3}$; and this force (by Cor. 1. of this Prop.) is the difference of the forces at V, by which the bodies revolve, P in the unmoved ellipse, VPK, and *p* in the moveable one, *upk*. And since (by this Prop.) this difference at any other altitude, A, is to itself at the altitude CV as $\frac{1}{A^3}$ to $\frac{1}{CV^3}$, its value for the altitude A is $\frac{R.G^2-R.F^2}{A^3}$. Hence, to the force $\frac{F^2}{A^2}$, with which the body may revolve in the immoveable ellipse VPK, let there be added the excess $\frac{R.G^2-R.F^2}{A^3}$, and the whole force shall be compounded $\frac{F^2}{A^2}+\frac{R.G^2-R.F^2}{A^3}$, with which the body can revolve in the moveable ellipse *upk* in the same time.

COR. 3. After the same manner it may be inferred, that if the immoveable orbit VPK be an ellipse, with its centre in the centre of force C, and there be described a moveable ellipse, *upk*, similar, equal, and concentric with it; and if 2R be the principal *latus rectum* of this ellipse, and 2T the *latus transversum* or axis major, and the angle VC*p* be always to the angle VCP as G^2 to F^2; the forces by which the bodies can revolve in equal times in the immoveable and moveable ellipses will be as $\frac{F^2A}{T^3}$ and $\frac{F^2.A}{T^3}+\frac{R.G^2-R.F^2}{A}$ respectively.

Cor. 4. And, generally, if the greatest altitude, CV, be named T, and the radius of curvature of the orbit at V, that is, the radius of the equi-curve circle be named R, and the centripetal force of the body in the immoveable trajectory in any place, V, be called $\frac{V.F^2}{T^2}$, and in other places, P, be indefinitely called X, the altitude CP being called A, and if G be taken to F in the given ratio of the angle VC*p* to the angle VCP: the centripetal force with which the same body can perform the same motions in the same trajectory, *upk*, revolving with a circular motion in the same times, is as the sum of the forces $X+\frac{V.R.G^2-V.R.F^2}{A^3}$.

Cor. 5' Being given the motion of a body in any immoveable orbit, its angular motion about the centre of force can be increased or diminished in a given ratio, and thus new immoveable orbits be found in which bodies would revolve with the new centripetal forces.

Cor. 6. If therefore to the right line CV given in position, there be erected a perpendicular, VP, of indeterminate length, and CP be joined, and C*p* drawn equal thereto, constituting the angle VC*p*, having to VCP a given ratio; the force with which the body revolves in the curve V*pk*, which the point *p* perpetually touches, will be reciprocally as the cube of the altitude C*p*. For the body P by its inertia, without the action of any other force, can uniformly describe the right line VP. Let there be added a force to the centre, C, reciprocally proportional to the cube of the altitude CP or C*p*, and (by what has been demonstrated) the rectilinear motion will be turned into a curve line V*pk*.

PROP. XLV.

To determine the motion of the apsides of orbits most nearly circular.

The problem is solved arithmetically, if the orbit which the body in the ellipse (as in the second or third Cor. of the preceding) describes in an immoveable plane be made to approach to the form of that in which the motion of the apsides is required, and by seeking the apsides of the orbit which that body describes in an immoveable plane. But the orbits will acquire the same form, if the centripetal forces with which they are described, when compared, be made proportional at equal altitudes. Let V be the higher apse, and let there be written, T for the greatest altitude, CV, A for any other altitude, CP or C*p*, and X for the difference of the altitudes CV—CP; and the force with which the body moves in the ellipse revolving about its focus C, (as in Cor. 2.,) and which was as $\frac{F^2}{A^2} + \frac{RG^2 - RF^2}{A^3}$, that is, as $\frac{F^2A + RG^2 - RF^2}{A^3}$, by substituting T—X for A will be as $\frac{R.G^2 - R.F^2 + T.F^2 - F^2.X}{A^3}$. Any other centripetal force is similarly to be reduced to a fraction whose denominator is A^3, and the numerators compared by a comparison of homologous terms. The matter will appear by examples.

Ex. 1. Let us suppose the centripetal force uniform, and therefore as $\frac{A^3}{A^3}$, or (writing T—X for A in the numerator,) as $\frac{T^3 - 3T^2.X + 3T.X^2 - X^3}{A^3}$; and comparing the corresponding terms of the numerators, namely, the given terms with the given, and the unknown with the unknown, it will be $R.G^2 - R.F^2 + T.F^2$ to T^3 as $-F^2.X$ to $-3T^2.X + 3T.X^2 - X^3$, or as $-F^2$ to $-3T^2 + 3T.X - X^2$. Now when the orbit is taken the nearest possible to a circle, let it actually coincide with a circle; and in consequence of the equality of R and T, and the indefinite diminution of X, the ultimate ratios will be $R.G^2$ to T^3 as $-F^2$ to $-3T^2$, or G^2 to T^2 as F^2 to $3T^2$, and *alternando* G^2 to F^2 as T^2 to $3T^2$, that is, as 1 to 3; and therefore G to F, that is, the angle VCp to the angle VCP as 1 to $\sqrt{3}$. Hence, when the body in the immoveable ellipse, by descending from the higher to the lower apside, would describe an angle (if it may be so called) VCP of 180 degrees; another body in the moveable ellipse, and therefore in the immoveable orbit, of which we are treating, will describe in its descent from the higher to the lower apside an angle of $\frac{180}{\sqrt{3}}$ degrees; and this because of the similitude of the orbit described with the uniform centripetal force; and the orbit which the body in the moveable ellipse describes in the quiescent plane. By the preceding comparison of terms the orbits are made similar, not generally, but when they approach most nearly to the circular form. A body, therefore, revolving with an uniform centripetal force in an orbit nearly circular, will, between the highest and the lowest apside, continually describe an angle of $\frac{180}{\sqrt{3}}$ degrees, or 103° 55′ 23″ at the centre; passing from the highest to the lowest apside after once describing this angle, and thence returning to the highest after a second time describing the same angle, and so on *ad infinitum*.

Ex. 2. Let us suppose the centripetal force to be as any power, A^{n-3}, of the altitude A, or $\frac{A^n}{A^3}$; where $n-3$ and n

signify any indices, integral or fractional, rational or irrational, affirmative or negative. That numerator, A^n or $(T-X)^n$, reduced to an indeterminate series by our method of converging series, becomes $T^n - n.T^{n-1}.X + \frac{n^2-n}{2}.T^{n-2}.X^2 -$&c. And comparing its terms with the terms of the other numerator, $R.G^2 - R.F^2 + T.F^2 - F^2.X$, there is $R.G^2 - R.F^2 + T.F^2$ to T^n as $-F^2$ to $-nT^{n-1} + \frac{n^2-n}{2}.T^{n-2}.X^2 -$&c. And by taking the ultimate ratios, when the orbits approach the circular form, there is $R.G^2$ to T^n as $-F^2$ to $-nT^{n-1}$, or G^2 to T^{n-1} as F^2 to $n.T^{n-1}$, and *alternando* G^2 to F^2 as T^{n-1} to $n.T^{n-1}$, that is, as 1 to n; and therefore G to F, that is, the angle VCp to the angle VCP as 1 to $\sqrt{n}$. Hence, since the angle VCP, described in moving from the higher to the lower apside in the ellipse, is of 180 degrees, there will be described the angle VCp in the descent from the higher to the lower apside, in the nearly circular orbit which a body describes with a centripetal force proportional to A^{n-3}, equal to $\frac{180}{\sqrt{n}}$ degrees; and repeating this angle, the body will return from the lower to the higher apside, and so on *ad infinitum*. As, if the centripetal force be as the distance of the body from the centre, that is, as A or $\frac{A^4}{A^3}$, n will be equal to 4, and $\sqrt{n}$ equal to 2; and consequently the angle between the higher and the lower apside equal to $\frac{180}{2}$, or 90 degrees. After completing the fourth of one revolution the body will arrive at the lower apside; after another fourth, at the higher; and so on continually. This also appears from Prop. 10. For the body urged by this centripetal force will revolve in an immoveable ellipse, the centre of which is in the centre of force. But if the centripetal force be reciprocally as the distance, that is, directly as $\frac{1}{A}$ or $\frac{A^2}{A^3}$, n shall equal 2, and therefore the angle between the higher and lower apse will be $\frac{180}{\sqrt{2}}$ degrees, or 127° 16′ 45″, and consequently a body revolving with such a force, by the perpetual repetition of this angle, will pass for ever from the higher to the lower, and from the lower to the higher apside alternately. Moreover, if the force be reciprocally as the fourth root of the eleventh power of the distance, that is, reciprocally as $A^{\frac{11}{4}}$, and therefore directly as $\frac{1}{A^{\frac{11}{4}}}$, or as $\frac{A^{\frac{1}{4}}}{A^3}$, n will equal $\frac{1}{4}$, and $\frac{180}{\sqrt{n}}$ equal to 360°, and consequently the body setting out from the higher apside, and thenceforward perpetually descending, will arrive at the lower apside after completing an entire revolution; after that in perpetual ascent, by completing another entire revolution, it will return to the higher apside, and so alternately for ever.

Ex. 3. Assuming m and n for any indices of powers of the altitude, and bc for any given numbers, let us suppose the centripetal force to be as $\frac{b.A^m + c.A^n}{A^3}$, that is, as $\frac{b.(T-X)^m + c.(T-X)^n}{A^3}$, (or by our same method of converging series,) as

$$\frac{b.T^m + c.T^n - mb.X.T^{m-1} - nc.X.T^{n-1} + \frac{m^2-m}{2} \cdot b.X^2.T^{m-2} + \&c.}{A^3}$$

and comparing the terms of the numerators, it will be $R.G^2 - R.F^2 + T.F^2$ to $b.T^m + c.T^n$ as $-F^2$ to $-mb.T^{m-1} - nc.T^{n-1} + \frac{m^2-m}{2} \cdot b.XT^{m-2} + \frac{n^2-n}{2} \cdot c.X.T^{n-2}$ &c. And by taking the ultimate ratios which arise when the orbits approach the circular form, it becomes G^2 to $b.T^{m-1}+cT^{n-1}$ as F^2 to $mbT^{m-1}+ncT^{n-1}$, and *alternando* G^2 to F^2 as $b.T^{m-1}+c.T^{n-1}$ to $mb.T^{n-1}+nc.T^{n-1}$, which proportion, by expounding the greatest altitude, CV or T, arithmetically by unity, becomes G^2 to F^2 as $b+c$ to $mb+nc$, and therefore as 1 to $\frac{mb+nc}{b+c}$. Whence G is to F, that is, the angle VCp to the angle VCP, as 1 to $\frac{mb+nc}{b+c}$. And consequently, since the angle VCP, between the higher and lower apside in the immoveable ellipse, amounts to 180 degrees, the angle VCp, between the same apsides in the orbit which the body describes with a centripetal force proportional to $\frac{b.A^m+c.A^n}{A^3}$, will be equal to $180.\sqrt{\frac{b+c}{mb+nc}}$ degrees. And, by the same argument, if the centripetal force be as $\frac{b.A^m-c.A^n}{A^3}$, the angle between the apsides will be found to be $180.\sqrt{\frac{b-c}{mb-nc}}$ degrees. After the same manner the problem is resolved in more difficult cases. The quantity to which the centripetal force is proportional ought to be resolved into converging series, having for a denominator A^3. Then the given part of the resulting numerator, and the part not given, as also the given part of the numerator $R.G^2-R.F^2+T.F^2-F^2.X$, and the part not given, are to be set down as having the same ratio; and by erasing the superfluous quantities, and writing unity for T, we shall obtain the proportion of G to F.

COR. 1. Hence, if the centripetal force be as any power of the altitude, this power can be found from the motion of the apsides, and conversely. If, for instance, the whole angular motion with which the body returns to the same apse, be to the angular motion of a single revolution, or of

360°, as some number, m, to another, n, and the altitude be called A: the force will be as that power of the altitude $A^{\frac{n^2}{m^2}-3}$, having for its index $\frac{n^2}{m^2}-3$. As is obvious by the second examples. Hence it appears, that this force cannot decrease in a ratio higher than the triplicate ratio of the altitude, in receding from the centre. A body revolving with such a force, and departing from an apse, if it begin to descend, will never arrive at the lower apse or least distance, but will descend perpetually to the centre, describing that curve of which we have treated in Cor. 6., Prop. 29. But if it begin at departing from an apse, to ascend even in the least degree, it will ascend *in infinitum*, and will never arrive at the highest apse. For it will describe that curve of which we have treated in the same Cor. So also when the force, as we recede from the centre, decreases in a ratio higher than the triplicate ratio of the altitude, according as it begins to descend or ascend, so it will continue either into the centre or off to infinity. But if the force, as we recede from the centre, decrease in a ratio less than the triplicate ratio of the altitude, or increase in any ratio of the altitude, the body will never continue to descend to the centre, but will at length arrive at the lowsst apse; and conversely, if a body descending and ascending alternately from apse to apse never arrive at the centre, the force, as we recede from the centre, will either increase, or will diminish in a ratio less than the triplicate ratio of the altitude: and the sooner the body returns from apse to apse, the more will the ratio differ from the triplicate. So that if the body in 8, or 4, or 2, or 1½ revolutions from the highest apse, return again to the same apse, first descending and then ascending; that is, if m be to n as 8, or 4, or 2, or 1½ to 1, and therefore $\frac{n^2}{m^2}-3$ be equal to $\frac{1}{64}-3$, or $\frac{1}{16}-3$, or $\frac{1}{4}-3$, or $\frac{4}{9}-3$, the force will be as $A^{\frac{1}{64}-3}$, or $A^{\frac{1}{16}-3}$, or $A^{\frac{1}{4}-3}$, or $A^{\frac{4}{9}-3}$, that is, reciprocally as $A^{3-\frac{1}{64}}$, or $A^{3-\frac{1}{16}}$, or $A^{3-\frac{1}{4}}$, or $A^{3-\frac{4}{9}}$. If the body in its several revolutions return to the same apse remaining unmoved, m will be to n as 1 to 1, and consequently $A^{\frac{n^2}{m^2}-3}$ equal to A^{-2} or $\frac{1}{A^2}$, and therefore the force will diminish in the duplicate ratio of the altitude, as has been demonstrated in the preceding. If the body in three-fourths of a single revolution, or two-thirds, or one-third, or one-fourth, return to the same apse, m will be to n as $\frac{3}{4}$, or $\frac{2}{3}$, or $\frac{1}{3}$, or $\frac{1}{4}$ to 1, and therefore $A^{\frac{n^2}{m^2}-3}$ equal to $A^{\frac{11}{9}-3}$, or $A^{\frac{9}{4}-3}$, or A^{9-3}, or A^{16-3}; and consequently the force will be reciprocally as $A^{\frac{11}{9}}$ or $A^{\frac{3}{4}}$, or directly as A^6 or A^{13}. Finally, if the body in proceeding from the highest to the lowest apside complete an entire revolution, and three degrees over, and that, therefore, that apside in the several revolutions of the body advances three degrees *in consequentia*, m will be to n as 363° to 360°, or as 121 to 120, and therefore $A^{\frac{n^2}{m^2}-3}$ will be equal to $A^{-\frac{29523}{14641}}$, and consequently the centripetal force will be reciprocally as $A^{\frac{29523}{14641}}$, or reciprocally as $A^{2\frac{4}{243}}$, nearly. The centripetal force consequently decreases in a ratio somewhat greater than the duplicate, but one which approaches $59\frac{3}{4}$ times nearer to the duplicate than the triplicate.

COR. 2. Hence, also, if a body with a centripetal force, reciprocally as the square of the altitude, revolve in an ellipse, having its focus in the centre of force, and there be added or subtracted any extraneous force; the motion of the apsides resulting from that extraneous force can be discovered, and conversely. If, for instance, the force in the ellipse be $\frac{1}{A^2}$, and there be an extraneous *ablatitious* force, $c.A$, and that, therefore, the remaining force be $\frac{A-c.A}{A^3}$; we shall have in the third examples b equal 1, m equal 1, and n equal 4; and consequently, the angle of revolution between the apsides equal to the angle $180°.\sqrt{\frac{1-c}{1-4c}}$. Let us suppose that extraneous force to be 357.45 parts less than that other force wherewith the body rcvolves in the ellipse, that is, c to be $\frac{100}{35745}$, A or T being equal to 1, and $180.\sqrt{\frac{1-c}{1-4c}}$ will become $180.\sqrt{\frac{35645}{35345}}$, or 180°.7623, that is, 180° 45′ 44″. Therefore, the body setting out from the highest apse, with an angular motion of 180° 45′ 44″, will arrive at the lowest apse, and will return to the highest apse after doubling this motion; and consequently, the highest apse in its several revolutions will advance 1° 31′ 28″. The lunar apse is about double as quick.

NOTES

TO

PART I.

NOTE (1). DEF. I.—*As I have found by experiments, &c.* Newton, in the sixth section of his second book, proves, that the quantities of matter in pendulous bodies, whose centres of oscillation are equally distant from their points of suspension, are in a ratio compounded of the ratios of the weights and times of oscillations *in vacuo.* Now observation proves, that the times are equal *in the same place* when the lengths of the pendulums are equal, and that, therefore, the quantities of matter are ever proportional to the weights.

(2.) DEF. III.—*The moon also, either by the force of gravity, &c.* The method here alluded to, of proving that the moon is retained in her orbit by the action of terrestrial gravitation, varying inversely as the square of the distance, is the same in principle as that given in Art. (238). In practice, however, there is some difference, as in Art. (238.) we computed the moon's parallax, and compared it with observation; but here, the thing computed is the space fallen through at the earth's surface in a second. This space is $\frac{2r\pi^2}{\Pi^3 . t^3} \cdot (1 + \frac{1}{358}) \left(\frac{\oplus}{\oplus + ☽}\right)$ by computation. If this space be observed on the parallel of

54° 44′ to be s, we must multiply it by $(1+\frac{1}{432})$, to allow for centrifugal force in this latitude; then comparing these values they are found nearly to agree. By the seconds pendulum s is found.

(3.) Cor. 4, *of the laws.* Let AC, BD (Fig. 51.) be spaces passed over in the same time, by two bodies moving uniformly along right lines in the same plane; and let the line CD, joining the bodies, be cut in a given ratio in K; the locus of K is a right line: for, take BG : AE :: BD : AC, then the ratio of CE to GD is given; take EF = GD, and join CF, then the triangle CEF is given in species; draw KL parallel to FD, and join EL, and since the ratio of EF to FC is given, and also the ratio of CF to FL, the triangle EFL is given in species, and the line EL is given in position; and since FD(= EG) is given in magnitude, LK is given in magnitude; and taking EH = LK, the line HK is given in position, and this line is the locus required.

If the two points do not move in the same plane, their motions being projected perpendicularly on *any plane whatsoever*, and the line joining the points projected being divided inversely as the bodies, the dividing point, which is the projection of the centre of gravity, shall move in a right line, and therefore the centre of gravity itself.

The same reasoning may obviously be extended to any number of bodies moving in right lines.

SECTION I.

(4.) *Lemma* I. Quantities are said to tend to equality, whose difference so varies as to have to the quantities themselves a ratio less and less continually. Quantities

The method of limits may be considered as reducible to the following heads. 1°. The definition of a limit. 2°. Problems by which limits so defined are obtained. 3° Properties of those limits. 4°. Application of those properties to geometrical demonstrations, to algebraical processes, and to the translation of physical relations into mathematical ones.

I. A limit of a variable quantity, whether a simple magnitude or the magnitude of a ratio, may be defined to be that quantity to which the variable quantity may so approach, as to differ from it in magnitude by less than any assigned or stated quantity.

Ex. 1^m. A variable secant from a given point, without a circle, may be so drawn as to differ from the tangent in magnitude, by less than any assignable magnitude. For, assign a difference, then the position of the secant is determined; and therefore, between the secant and tangent another secant can be drawn. The magnitude of the tangent is a *limit* of the magnitude of the secant, but the secant is not said to become the tangent.

2^m. A limit of the quantity $\frac{x^2-a^2}{x-a}$, when x is greater than a, is $2a$; for they may be made to differ by less than any assignable magnitude. But $2a$ cannot, with propriety, be said to be one of the values of $\frac{x^2-a^2}{x-a}$, or to be equal to $\frac{0}{0}$.

3^m. The limiting ratio of the increments of the abscissa and ordinate of a curve, is the ratio of the subtangent to the ordinate. The ratio of the increments themselves is never equal to the ratio of the subtangent to the ordinate.

II. It is then a mathematical problem to find a limit so defined, when the variable quantity admits of one. The solution of the problem is evidently had, if the variable are implicitly defined to be those in which series of parallelograms are inscriptable, equal in number, and to each other in the same ratio. Assuming this definition as the basis, it is easy to prove, that all right lines similarly posited in two such curves, are to each other in the same ratio, that the areas are in the duplicate ratio of the homologous sides, that the curves may be so placed, that any two coinciding *radii vectores* from the same point would be to each other in a constant ratio, and that their equations may be so simplified as to involve but one linear constant.

This last property is very frequently made the definition, and it serves best for this purpose in an analytical point of view.

(7.) *Lemma* VI, VII, VIII, *and* IX. The understanding of these lemmas cannot occasion the slightest difficulty to any person acquainted with the theory of contact. From the difficulty, or rather the absurdity, of attempting to contemplate quantities which elude the notice of our senses, Newton has had recourse to the artifice of substituting for these, finite quantities which are ever to each other in the same ratio. It may be observed, with respect to the eighth lemma, that the line RBD is understood to move parallel to itself, this appears from the words "*triangula tria semper finita*" in the text.

On the method of prime and ultimate ratios.

(8.) A more excellent commentary on this subject could not be given than is contained in the following remarks and examples, taken from a paper by Dr. Brinkley, published in the Transactions of the Royal Irish Academy for 1802.

The method of limits may be considered as reducible to the following heads. 1°. The definition of a limit. 2°. Problems by which limits so defined are obtained. 3° Properties of those limits. 4°. Application of those properties to geometrical demonstrations, to algebraical processes, and to the translation of physical relations into mathematical ones.

I. A limit of a variable quantity, whether a simple magnitude or the magnitude of a ratio, may be defined to be that quantity to which the variable quantity may so approach, as to differ from it in magnitude by less than any assigned or stated quantity.

Ex. 1^{m}. A variable secant from a given point, without a circle, may be so drawn as to differ from the tangent in magnitude, by less than any assignable magnitude. For, assign a difference, then the position of the secant is determined; and therefore, between the secant and tangent another secant can be drawn. The magnitude of the tangent is a *limit* of the magnitude of the secant, but the secant is not said to become the tangent.

2^{m}. A limit of the quantity $\frac{x^2-a^2}{x-a}$, when x is greater than a, is $2a$; for they may be made to differ by less than any assignable magnitude. But $2a$ cannot, with propriety, be said to be one of the values of $\frac{x^2-a^2}{x-a}$, or to be equal to $\frac{0}{0}$.

3^{m}. The limiting ratio of the increments of the abscissa and ordinate of a curve, is the ratio of the subtangent to the ordinate. The ratio of the increments themselves is never equal to the ratio of the subtangent to the ordinate.

II. It is then a mathematical problem to find a limit so defined, when the variable quantity admits of one. The solution of the problem is evidently had, if the variable quantity or its equal can be expressed as the sum or difference of two quantities, one fixed, and the other variable; the latter admitting a value less than any assigned. The fixed one is the limit.

Supposing x greater than a, and a limit of $\frac{x^2-a^2}{x-a}$ be required. Since $\frac{x^2-a^2}{x-a}=x+a=2a+e$, (putting $x=a+e$), and since e may be any assigned magnitude, $2a$ is the limit.

To determine the limiting ratio of the incr. of the abscissa to the incr. of the ordinate, we have incr. of abs. : incr. of ord. :: subt. $+v$: ord. by sim. triangles. As the ordinates may approach, so that v may become $<$ any assigned magnitude, the limit of the latter, and therefore, of the former ratio, is the ratio of the subtangent to the ordinate. This method is frequently applicable when the terms of the proposed ratio are both variable. A ratio is found equal to the given ratio, one of the terms of which is fixed, and the other variable. A limit of the variable is then found, and thence the limit of the proposed ratio is had.

Thus it is that Newton investigates the limiting ratio of the arc to the chord, tangent, &c. In this way, also, the limiting ratio of the increments of the algebraical quantities x and x^n is had. Let o be the increment of x, and $n.x^{n-1}.o+P.o^2+$&c. is the increment of x^n; the general ratio of those increments is $1 : n.x^{n-1}+Po+$&c. In this series it may readily be shewn that o may be taken so small, that the sum of the terms after the first may be less than any quantity that can be assigned, so that according to the definition, $1 : n.x^{n-1}$ is the limiting ratio required.

There is no shifting of the question, as it has been termed, by first taking o a real magnitude, and then no magnitude, nor any introduction of infinitesimal quantities. The question is concerning the limit of the ratio of the increments not concerning the ratio itself; the general ratio of the increments, is equivalent to a ratio, one term of which is fixed, and the other composed of a fixed and variable quantity; this variable quantity being susceptible of a less value than any that can be assigned, the ratio of the fixed quantities is, by the definition, the limiting ratio.

For the ready solution of this problem, the following proposition is often of the greatest importance, and is one of the principal sources of compendium derived from this doctrine.

Prop. A. *In deducing a limit of a variable quantity, depending upon other variable quantities, the limits of these may be used for the quantities themselves in any part of the process.*

Let L be the limit of any variable quantity, M, involved in the process; then in any of the steps L may be used for M, and the conclusion will be true. For let $e=M-L$ express the general relation between L and M, then, without considering the limit, in the conclusion will be found $L+e$ instead of L. But to obtain the limit required, the limits of the variable quantities must be used, and therefore, L must be substituted for $L+e$. This proposition relates both to geometrical magnitudes and analytical quantities. Any steps which we can demonstrate will lead to a true conclusion, must be considered, with reference to that conclusion, as logical. Such are the steps in which we substitute limits for the quantities themselves, although the quantities are never accurately equal to their limits. By subjoining the word *ultimo* to such steps, we refer to the general conclusion, and intend these steps as only true with a reference to that. From being thus enabled to shorten the steps of a demonstration, arises much of the value of the method, whether it be applied to geometrical demonstration or analytical processes. This

Prop. is assumed in the 3d Cor. of 7th, and in Cor. 8th of Newton's lemmas.

III. From the properties of those limits, the three following may be selected as frequently used; of which the second, as being more connected with the above theorem, is particularly considered.

1°. *Quantities which are the same limits of the same variable quantity, are equal to one another.*

2°. Prop. B. *If the limiting ratio of the corresponding increments of two magnitudes commencing together be always a ratio of equality, the magnitudes themselves are in a ratio of equality.*

For, let M and N be the two magnitudes, and let M be divided into *any* number, n, of equal parts, p; and let also N be divided into the same number of corresponding parts, q, r, s, t, &c. Then the parts of N are all equal to each other, or some of them are unequal.

If they be always all equal, $M : N :: p : q$, and the limiting ratio of the increments is the same as the constant ratio of the increments themselves, and therefore $M : N$ is a ratio of equality.

If the parts of N be not equal to each other, one of them must be greatest and one least; therefore, taking n *any* number, if q be greatest and r least, $nq > N$ and $nr < N$. If $M : N$ be not a ratio of equality, let it differ from the ratio of equality by the ratio $e : f$. Now the ratio $M : N$ is between the ratios $np : nq$ or $p : q$, and $np : nr$ or $p : r$, consequently, one of them must differ from the ratio of equality by a ratio $< e : f$; but by hypothesis, the ratios $p : q$, $p : r$, &c. may be nearer the ratio of equality than by any assigned ratio. Hence, the ratio $M : N$ cannot differ from a ratio of equality.—*See Cor. Lemma* 4.

3°. *If an equation exist between the corresponding variable quantities* $L + e$, $L' + e'$, &c. *where* e, e', &c. *may be* $<$ *any assigned magnitudes, however small,* (L, L' &c. *being fixed, and therefore the limits of those quantities*)*; then, if in that* *equation* L, L′, &c. *be substituted for* $L+e$, $L'+e'$, &c., *an equation will be had between the limits.*

IV. The following instances will serve for illustrating the application of these principles.

1°. As a geometrical instance. Dr. Brinkley, in the paper whence these extracts are made, reduces the finding of the surface of an oblique cylinder to the rectification of an ellipse, by proving the following theorem. *The surface of an oblique cylinder is equal to a rectangle contained by the diameter of its base and the circumference of an ellipse, the axes of which are, the length and perpendicular height of the cylinder.* In order to effect this rectification, he uses a theorem of Fagnani's on elliptic arcs, which theorem he gives a new demonstration of, by the method of prime and ultimate ratios, founded on a new property of the ellipse, which he premises as a lemma. The property is as follows:—*Let* AED (Fig. 52.) *be a semi-ellipse on the axis major*, AD, C *the centre, and* AFD *the circumscribing circle. Let also any ordinate*, GE, *be produced to meet the circle in* F. *Draw* FC *intersecting the ellipse in* O, *and* QOP *parallel to* FG; *then, if* QC *be drawn intersecting the tangent* EN *in* N, CN *will be perpendicular to the tangent* EN *and equal to* CO. It also follows from this, that *when* CO *is a mean proportional between* AC *and* CB, CE = *semi-conjugate to* CO, *and also* $EN = AC - CB$. The proof of these we shall leave to the reader. Fagnani's theorem is as follows:—

Theorem. AEOBD (Fig. 53.) *is an ellipse, the axis major of which is* AD. *Produce any ordinate*, GE, *to meet the circumference of the circle circumscribing circle in* F. *Draw* COF *and* CT *a semi-conjugate to* CO, *and also the tangent* TM *meeting the perpendicular* CM *in* M; *then arc* BT — *arc* AE = TM.

Draw fg indefinitely near to FG. Let Ct be semi-conjugate to Co, and draw TR, tr parallel to the axis minor BC. Let also CR, Cr meet the ellipse in L, l. Draw

LK, lk parallel to the axis minor, join KM, kC, and draw the tangent tm meeting kC, which intersects TM in v. Then by the lemma it easily appears, that K, M, C are in one right line. Because FCK, fCk are right angles, $Ff = Kk$. Also, if Fn be parallel to the tangent at E, the triangle Ffn is ultimately similar to CLK; for, CL is parallel to the tangent at E, because FCK is a right angle, and therefore, CE and CL are conjugate to each other; and also, CK is parallel to a tangent at F. Therefore, *ultimo* $Ff = Kk : Fw :: CK : CL = CM$ by lemma. Also, *ultimo* Mv[*] $: Kk : CM : CK$, consequently, *ultimo* $Mv = Fw$. But *ultimo* $Mv = TM - tm + \text{arc } Tt$ and $Fw = Ee$, therefore, *ultimo* $Tt - Ee = tm - TM$, or *ultimo* the incr. of BT — incr. of AE = incr. of tang. MT. Therefore, as these magnitudes begin together, arc BT — arc $AE = TM$.

Cor. If Co be taken a mean proportional between AC and BC, and be produced to meet the circle in F, then drawing the ordinate FG, the point of intersection will divide the elliptic quadrant, so that $BE - AE = AC - BC$. For, by the construction CE = semiconj. to CO, (by lemma), and therefore $CE = CT$, and therefore $BE = BT$. Whence, TM = tang. at E = (by Cor. lemma) $AC - BC$. Consequently by the theorem, arc BE — arc $AE = AC - BC$.

In the demonstration of the above theorem, the *limiting* ratio of the incr. of the elliptic arcs to the increment of the tangent is deduced by the principles of Prop. A. From the limiting ratio of these increments, the ratio of the difference of the arcs to the tangent is deduced by

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[*] Let the tangents TM, tm intersect in n. Then $Mv = MT - vT = MT + Tn - vn$. But *ultimo* $vn = mn = tm - tn$, therefore *ultimo* $Mv = TM - tm + Tn + nt =$ *ultimo* $TM - tm + \text{arc } Tt$.

Prop. B, not from the ratio of the evanescent increments, but from the limiting ratio of the increments.

2°. The following method of finding the sine in terms of the arc, shews the application in an algebraic process. Let s = sine of arc a to rad. 1, and let sin. $\frac{a}{n} = \frac{a}{n} - e$, n representing any assigned number, however great. Then by a well known theorem for the sine of a multiple arc, $s = n.\left(\frac{a}{n} - e\right) - \frac{n.(n^2-1)}{2.3} \cdot \left(\frac{a}{n} - e\right)^3$ &c. $= \overline{a - ne} - \left(1 - \frac{1}{n^2}\right)\frac{(a-de)^3}{2.3}$ + &c. Now the limit of $a - ne$ is a, for the limiting ratio of $\frac{a}{n} : \frac{a}{n} - e$ (limiting ratio of arc to sine) is the ratio of equality, or of $a : a - ne$, therefore the limit of $a - ne$ is a. Hence, taking the limits of the terms of the above series, we have $s = a - \frac{a^3}{2.3}$ + &c. This equation is not deduced by neglecting quantities as infinitely small, but because the first equation is true when n is any assigned number, however great; the last must necessarily be true also, for otherwise it is easily shewn, that the first could not be generally true when n is any assigned number, however great.

3°. The ratio of the centripetal forces in two points of a curve is the limiting ratio of the sagittæ. The limiting ratio of the sagittæ in the ellipse, the force tending to the focus, is the inverse duplicate ratio of the distances. And as the ratios, which are the limits of the same variable ratio, must be equal to each other, the ratio of the forces is the inverse duplicate ratio of the distances. The application of limits to physical enquiries strikingly illustrate the value of the method. The reader will meet with so many beautiful instances of this kind in the text, that it is unnecessary to dwell more upon them here.

If the reader wish to become thoroughly acquainted with the doctrine of limits, he will find the most splendid applications of it constantly occurring all through the papers of the great mathematician to whom the above observations are due. More particularly in a paper in the Irish Transactions for 1798, in which he gives his method of finding fluxions *per saltum*; and another paper in 1817, in which he refutes an apparent objection against this doctrine that occurs in the integration of certain differential equations; and a second objection, urged by Lagrange, in a case of finite differences. The extent and the power of the doctrine of limits, when rightly applied, can only be conceived by a person who has studied the papers of Dr. Brinkley.

SECTION II.

(8.) After the same manner it may be demonstrated, that a body acted on by two fixed centres of force will describe equal solids in equal times, having a common edge line; namely, the line joining the centres of force. This readily follows from the fact, that pyramids of equal bases and altitudes are equal.

(9.) *Cor.* 1, *Prop.* I. The velocities in different points of different orbits are directly as the synchronous areas, and inversely as the perpendiculars on the tangents; and the angular velocities directly as the synchronous areas, and inversely as the squares of the distances.

(10.) *Cor.* 5, *Prop.* I. By this Cor. we shall be able to convert the variations given in the succeeding Props. into absolute values.

(11.) *Prop.* II. There are two cases in which the centre of equal areas does not indicate a centre of force, viz. when a single body moves in consequence of a primitive impulse, and when the only forces acting on the bodies of a system are their mutual attractions. This is shewn in the second Part of this Treatise.

(12.) *Prop.* IV. The variations in this Prop. may be converted into equations as follows:—Let f denote the force required, and g the force of gravity; let a denote the radius of the circle, and m the sagitta of the parabolic arc described in the same time as the circular arc; then, by this Prop. and Cor. 5, of Prop. I., we have $f : g :: \frac{(\text{arc})^2}{2a} : m$, whence $f = \frac{g(\text{arc})^2}{2am}$. If f and g do not mean abstract numbers expressing the ratio of the forces, but effects produced thereby: then by the second law of motion f and g denote velocities acquired in the same time, and consequently $g = 2m$. Hence we have $f = \frac{(\text{arc})^2}{a}$. From Cor. 1, we have $f = \frac{g \cdot v^2}{2am}$ or $f = \frac{v^2}{a}$, according to the different significations of f.

In Cor. 2, let t denote the periodic time in seconds, then the motion being uniform, we have $2\pi a : v :: t : 1''$, therefore $v = \frac{2\pi a}{t}$, and consequently $f = \frac{2\pi^2 ag}{mt^2}$ or $f = \frac{4\pi^2 a}{t^2}$.

(13.) *Cor.* 8, *Prop.* IV. This Prop. is extended to bodies describing *similar* parts of *similar* curves about centres of force *similarly* situated therein, as follows. The forces in two such curves are as the *sagittæ* that is by Lemma XI, as the squares of the arcs divided by the chords of curvature, but from the similarity of the figures the chords of curvature are as the distances, therefore, &c.

The times of describing similar sectors are as the sectors directly and inversely as the synchronous areas; but the sectors are as the squares of the distances, and the synchronous areas as the distances into the velocities, and consequently the times are as the distances divided by the velocities, therefore, &c.

(14.) This Prop. is of some use in comparing the forces in different points of the same orbit, and thus discovering the law of force; its chief uses are to compare forces in different orbits when they are similar with the centres of force similarly situated, and to discover the periodic time or velocity in a circle uniformly described. when we know the force by some other means: *e. g.* for the conical pendulum, its height being h, and the distance from the vertical a, we have $f=\frac{g.a}{h}$, but also $f=\frac{4\pi^2 a}{t^2}$, equating these we find $t=2\pi\sqrt{\frac{h}{g}}$. Again, if a body revolve on the inside of a vessel, the sections of which, parallel to the horizon, are circles, the time of describing one of these sections may be found, for it we have $t=2\pi\sqrt{\frac{s}{g}}$ where s is the length of the subnormal; it appears hence, that if it be a paraboloid, the times for all sections are the same, and from this it may be deduced, that when a vessel full of water revolves about a vertical axis, the water within will assume a paraboloidal shape. We may also, by means of this Prop., compare the centrifugal force at the equator with the force of gravity.

(15.) The angular velocities in circles being inversely as the periodic times, and the synchronous areas as the velocities into the distances, the reader may exercise himself in considering how the angular velocities and synchronous areas are affected by different suppositions relative to the force.

(16.) *Cor.* 9. The force being given, the spaces are as the squares of the times, and the motion being uniform, the times are as the arcs, consequently, any spaces descended as the squares of the arcs; but if the space be indefinitely small, the square of the arc is equal to the rectangle under the diameter and space descended, consequently the same is true of any arc.

(17.) *Scholium.* The sixth Cor. applies in the case of the celestial bodies, for these have been observed to describe equal areas in equal times, and to have the squares of the times as the cubes of the distances.

The length of perimeter between any two angles of a regular polygon given in species, is proportional to the number of intervening angles into the radius of the circumscribing circle, therefore the number of angles is directly as this length, and inversely as the radius, therefore, &c.

(18.) Several simple problems might be proposed, the solutions of which would follow from this Prop.: *e. g.* given the velocity and absolute force, to determine the circle and time; given the circle and absolute force, to determine the requisite velocity and time, &c.

(19.) *Prop.* VI. The variations in this proposition may be easily converted into absolute values, as follows:—Let f denote the force in question, g the force of gravity, s the *sagitta* QR by the force f in the time t, and m the *sagitta* by the force of gravity in one second; then, $f : g :: \frac{s}{t^2} : m$, therefore, $f = \frac{gs}{mt^2}$. Or as in Note (12.), if f denote the effect produced, namely, the velocity acquired in the time t, then $f=\frac{2s}{t^2}$.

(20.) *Cor.* 1, *Prop.* VI. Let h denote double the area described in the unit of time, then $ht=\mathrm{SP}\times\mathrm{QT}$; hence $f=\frac{2h^2.s}{\mathrm{SP}^2\times\mathrm{QT}^2}$.

(21.) *Cor.* 2, *Prop.* VI. Also, $ht=\mathrm{SY}\times\mathrm{QP}$; hence $f=\frac{2h^2.s}{\mathrm{SY}^2\times\mathrm{PQ}^2}$.

(22.) *Cor.* 3, *Prop.* VI. Since $\mathrm{PV}=\frac{\mathrm{QP}^2}{\mathrm{QR}}$, we have $f=\frac{2h^2}{\mathrm{SY}^2\times\mathrm{PV}}$.

(23.) *Cor.* 4, *Prop.* VI. Since $\frac{h^2}{\mathrm{SY}^2}=v^2$, v denoting the velocity, we have $f=\frac{2v^2}{\mathrm{PV}}$. The results of the five last notes coincide exactly with the formulæ given in Arts. (54), (55), and (56) of the second part of this work.

(24.) Let γ denote the radius of curvature, then obviously $\mathrm{PV}=\frac{2\gamma.\mathrm{SY}}{\mathrm{SP}}$, therefore we have $f=\frac{h^2.\mathrm{SP}}{\gamma.\mathrm{SY}^2}$. This expression is due to John Bernouilli.

(25.) *The same is easily demonstrated by Cor.* 4, *Lemma* X; for the *sagittæ* are the spaces which would be described in the very beginning of the motion by the action of the central forces, and hence the forces are directly as these *sagittæ*, and inversely as the squares of the times.

(26.) The formula $f=\frac{2s}{t^2}$ is the most general of all that have been given in these notes; as it is true independantly of Kepler's first law, and is indeed actually the same as the first of formulæ (6), Art. (49.) of the second Part of this Treatise. All the other formulæ involve this law, and apply to the determination of the forces for different points of the same or different orbits. As announced in the text of Newton, however, none of them apply to different orbits but $f=\frac{2v^2}{PV}$, unless h be the same in those different orbits.

The quantity $\frac{SP^2 \times QT^2}{QR}$ is a solid, being of three dimensions, and is always a finite quantity in curves of continuous curvature, for it is equal to $\frac{SY^2 \times QP^2}{QR}$, in which SY is finite, and $\frac{QP^2}{QP}$ the chord of curvature, and consequently finite.

(27.) *Prop.* VII. The exact value of the force in this orbit is found by substituting $\frac{AV^2}{SP^2 \times PV^3}$ for $\frac{QR}{SP^2 \times QT^2}$ in the value of f; hence $f=\frac{2h^2.AV^2}{SP^2 \times PV^3}$ for the eccentric circle in general; and for the circumferential $f=\frac{2h^2.AV^2}{SP^5}$.

It is somewhat remarkable, that the forces at opposite points in the eccentric circle are inversely as the squares of the distances, and consequently the law of force at the apsides of the planetary orbits being the inverse square of the distance; Huyghen's could not thence justly infer, as Laplace asserts, the law of force in nature.

The theory of the eccentric circles derives some interest from the fact, that Kepler once supposed the planetary orbits to be of this kind. That they are not follows readily; for if so, the velocities would be reciprocally as SP × PV, instead of reciprocally as SY.

(28.) *Cor.* 2, *Prop.* VII. This Cor. would not hold unless the periodic times were the same, for if h were not given, f would not vary as $SP^2 \times PV^3$ inversely. In the next Cor. it may be remarked, that the body would move in the circle of curvature, but that the law of force would not answer.

The student may exercise himself by applying to this orbit the inquiries, the principles of which have been laid down in Sect. III., Part II. of this Treatise.

(29.) *Prop.* VIII. To determine the actual value of the force when directed to an infinitely remote centre, we proceed as follows. The component of the velocity transverse to the common direction of the force is constant; since the accelerating force in that direction is cypher. Let us denote this velocity by v', then the value of t in the formula $\frac{2s}{t^2}$ is $\frac{QT}{v'}$, (QT being the element of the line parallel to which v' is measured); substituting this value in the same formula, we have $f=\frac{2v'^2.s}{QT^2}$. Applying this to the semicircle, as in the text, we have $f=\frac{v'^2.CP^2}{PM^3}$. The law of force in the semicircle may be inferred from that in the eccentric circle, for the distance from the centre of force may be deemed invariable, and we shall then have the force inversely as the cube of the chord, or of its half the ordinate.

(30.) In any conic section the force along the ordinates to any diameter may be proved to be inversely as the cube of the ordinate, as follows. In any conic section if Am mP be tangents, and AM a diameter, it may easily be proved, that $Am^2 : AM^2 :: \frac{1}{4}L'^2 : PM^2$ when L′ denotes the parameter to the diameter AM. Hence as follows; $2PM \times QR : RP^2 :: Am^2 : mP^2$, and RP^2 :

$QT^2 :: mP^2 : AM^2$, therefore, $2\,PM \times QR : QT^2 :: Am^2 : AM^2 :: \frac{1}{4}L'^2 : PM^2$, and consequently $\frac{QT^2}{QR} = \frac{2\,PM^3}{\frac{1}{4}L'^2}$, or $f = \frac{\frac{1}{4}L'^2 . v'^2}{PM^3}$. This proof is exactly similar to that given in the text for the semicircle, and deduced from a property of the ellipse, which the author does not remember to have seen elsewhere. The proof usually given, and which is to be found in Jebb's Excerpta, is altogether unlike Newton's.

(31.) The formula established in Note (29.) may be applied to several other cases; for instance, to find the laws of force along the ordinates to the base or axis of a cycloid, to the asymptote of an hyperbola or logarithmic curve, or to the diameter of the semi-cubical parabola.

(32.) *Prop.* IX. It may be proved as follows;—that the force is inversely as the third power of the distance in the logarithmic spiral. Since the chord of curvature is twice the distance, the velocity is equal to that in the equidistant circle, hence the synchronous areas are equal in these curves; but these areas are equal at all distances in the logarithmic spiral, therefore they are equal in all the circles, and consequently, by Prop. 4, the force is inversely as the cube of the distance.

(33.) If the reader investigate the laws of force in the curves whose equations are $r\omega = a$, $r = a\,(\varepsilon^{\omega} \pm \varepsilon^{-\omega})$, and $r = \frac{a}{\cos. m\omega}$, he will find them to be the inverse third power of the distance as well as the logarithmic spiral. This investigation he cannot effect without previously establishing the properties of these curves, which can only be done by the employment of the calculus, and consequently this subject belongs more properly to the second Part of this Treatise.

(34.) *Prop.* X. More compendiously as follows:—

$$Pv \times vG (= 2\,PC \times QR) : Qv^2 :: PC^2 : CD^2$$

$$Qv^2 : QT^2 :: PC^2 : PF^2$$

$$\therefore 2\,PC \times QR : QT^2 :: PC^4 : CD^2 \times PF^2 (= CA^2 \times CB^2).$$

Hence

$$\frac{QT^2 . PC^2}{QR} = \frac{CA^2 \times CB^2}{2 . PC}.$$

If the same construction be used for the central hyperbola, the same notation and demonstration will apply *verbatim.*

(35.) *Prop.* X. *Cor.* 1. From Cor. 4, Prop. 6 it is manifest, that if the value of the force and velocity at any point of an orbit be known, the curvature at that point is determinable. Hence, knowing the law of force, absolute force, direction of projection, velocity of projection, and initial distance, the orbit can be determined in the present instance; for the curvature, distance, and position of the tangent being known, the central ellipse is determined. If the curvature be negative, the orbit determined is a central hyperbola. If the chord of curvature be twice the distance, and the angle of projection right, the orbit will be a circle; if the angle be not right the point of projection will be the extremity of one of the equal conjugate diameters. If the angle be right, but the chord of curvature not equal to twice the distance, the point of projection is an apse; the more remote if the chord of curvature be less than twice the distance, the less remote, if greater.

(36.) *Prop.* X. *Cor.* 2. The times are equal in ellipses that have the same axis major, for the times in these ellipses are as the whole areas directly, and the synchronous areas inversely; the whole areas are as the *axes minores*, and the synchronous areas at the common vertex are as two coinciding ordinates, for the deflections from the tangent in the same time are equal, but coinciding ordinates are as the *axes minores*, therefore, &c.

(37.) *Prop.* X. *Scholium. And as in the circle or ellipse, &c.*—Let there be two curve lines, AD, A*y*, (Fig. 34.) having a common vertex, A, and a common abscissa, AC; and let the nature of the curves be such, that having drawn ordinates CD, C*y*, the ordinate CD should be to the ordinate C*y* in a given ratio, and that they should be inclined to AC in the same or constant angles. Let two bodies, D and *y*, revolve in these figures under the influence of any forces, and let their periodic times be equal; if any point, S, be taken in the axis of abscissæ, then of the forces which act upon D and *y*, the components directed along SD, S*y* shall be as those lines SD, S*y*. For, since the periodic times are equal, the synchronous areas shall be as the whole areas, that is, as the ordinates CD, C*y*; but the sectors ASD, AS*y* are obviously as the ordinates DC, *y*C, consequently these sectors are as the synchronous areas, and are therefore described in the same time; hence the bodies D, *y* setting out together from A always arrive together at the extremities of corresponding ordinates DC, *y*C. Hence the velocities parallel to AS are equal, and those along DC, *y*C are as the lines DC, *y*C; and the bodies being supposed equal, the same is true of the accelerating forces. If, therefore, SD be taken to represent the whole force directed towards S for the body D, S*y* must represent the same for the body *y*; hence these forces are as the lines SD, S*y*.

SECTION III.

(38.) *Prop.* XI. More compendiously as follows:—

$$L \times QR : L \times PV :: QR : PV :: AC : PC$$
$$L \times PV : GvP :: L : Gv$$
$$GvP : Qv^2 :: PC^2 : CD^2$$
$$Qv^2 : QT^2 :: CD^2 : CB^2$$

whence by compounding these ratios,

$$L \times QR : QT^2 :: AC \times L \times PC^2 \times CD^2 : PC \times Gv \times CD^2 \times CB^2,$$

that is, in consequence of the equality of $AC \times L$ and $2BC^2$,

$$L \times QR : QT^2 :: 2PC : Gv,$$

but $2PC = Gv$, whence

$$L \times QR = QT^2, \therefore \text{\&c.}$$

(39.) The exact value of the force in this orbit may be found by substituting this value for $QT^2 \div QR$ in the formula of Note (20). The same may be found by Cor. 3, Prop. 7, knowing the force about the centre.

The focal conic sections having been so fully discussed in Sections IV and V, Part 2, of this work, it is is needless to comment upon the remaining propositions of this section.

SECTION VII.

(40.) *Prop.* XXXII. In this proposition the determination of the space fallen through is reduced to the problem of drawing from the vertex of a semicircle, equilateral hyperbola, or parabola, a line to cut off a given area. This is a particular case of Kepler's problem, and much more simple than the general one. In the case of the semicircle it may readily be effected, by describing a cycloid whose axis is AB. It has been shewn that the two second constructions are also requisite, see Art. (28.), p. 103, of this Treatise. Some late writers erroneously omit those cases.

(41.) *Prop.* XXXIII. This proposition follows at once from the fact, that in the ellipse and hyperbola, vel.: vel. in equid. circle :: $\sqrt{2a-r}$: $\sqrt{a}$, see Art. (91.), p. 135, for $(2a-r)$ is the distance from the further vertex, and a the principle semi-diameter.

(42.) *Prop.* XXXIV. This Prop. also appears from formula (18), p. 124, and from this that the velocities in circles are inversely as the square roots of the distances.

(43.) *Prop.* XXXV. In this Prop. he shews how the problem of Prop. 32 may be effected. The demonstration consists of two parts: 1st, to prove that AC : SK :: CD.Cc : Dd.SY; and 2d, that $Cc.Dd = AC.Kk$; these being proved, we have immediately $Dd \times SY = SK.Kk$. He applies this theorem in the two next propositions, to determine the times of ascent or descent of a body falling or projected in a right line.

(44.) *Prop.* XXXVIII. Let the ellipse Ayx and quadrant ADE (Fig. 34.) have their axes coincident. Then the centre of force being at S, the time of describing Ay is to the time of describing Ayx as the elliptic sector ASy is to the elliptic quadrant ASx, that is, as the sector ASD to the quadrant ASE, or as the arc AD to the arc AE; hence the time of describing any elliptic arc varies as the semicircular arc with the same abscess. This proportion is true, however the breadth of the ellipse be narrowed, and therefore true of the limiting state, viz. the right lined motion.

The velocity in the ellipse at y varies as the semi-conjugate to Sy, that is, as the square root of the product of the focal distances. When the ellipse is narrowed indefinitely the foci are at the extremities of the axis major, and the point y comes to C; hence in this case the product of the focal distances becomes the square of CD, and consequently the velocity in the line AC varies as CD, the right sine of the arc AD.

(45.) *Prop.* XXXIX. The method of reasoning in this proposition is as follows. If the velocity be measured by the square root of the area, we shall have DF as $\frac{V.I}{DE}$. But DF, the force, must be as the increment of the velocity applied to the increment of the time, but the incr. of the time is as $\frac{DE}{V}$, therefore DF is as $\frac{I.V}{DE}$. This reasoning, the reader will observe, is not perfectly logical.

(46.) *Prop.* XL, *Cor.* 2. The result in this Cor. must have been deduced from Prop. 39, by effecting the proposed quadrature. This could only be effected by the integral calculus. He gives only the result, as the deduction would not harmonize with his other investigations.

SECTION XI.

(47.) *Prop.* LVIII. The reasoning in this Prop. is as follows:—if the orbits were described in the same time the forces would be as the distances; but the forces are equal, consequently the times are not equal. Again, it is shewn that a longer time, and also a greater velocity, are required for the case of one body about the other at rest, in the subduplicate ratio of the distances. This Prop. is of great importance, as it enables us, having theoretically established the conditions of the motion of one body about another at rest, to apply the results obtained to the case of both, revolving about their common centre of gravity.

(48.) *Prop.* LX. Let T denote the period of P about S, and T′ that of p about s, A the given axis major, and A′ the required, then

$$T : T' :: \sqrt{S} : \sqrt{S+P}$$

$$T' : T :: A^{\frac{3}{2}} : A'^{\frac{3}{2}}$$

$$\therefore A^3 : A'^3 :: S+P : S, \text{ or } A : A' :: S+P : m,$$

where m is the first of two means between S+P and S.

(49.) *Prop.* LXI. Let r, r' denote the lines SP, PC respectively; let $S.\phi(r)$ be the force by which P is attracted to S, then this force on P, expressed as a function of r', is $S.\phi\left(\frac{S+P}{S}\cdot r'\right)$. If we conceive this equal to $M.\phi(r')$, M will be a third body, which, placed at the centre of gravity, would produce the same action, and according to the same law.

(50.) *Prop.* LXII. If, for instance, the force be inversely as the square of the distance, the force directed to C will be $\frac{S^3}{(S+P)^2} \cdot \frac{1}{r'^2}$ for the body P, and for S this force will be $\frac{P^3}{(S+P)^2} \cdot \frac{1}{(r-r')^2}$. If there be no angular motion in the system originally, we can determine the right lined motion of P and S, one towards the other, by using the above expressions for the force in Prop. 2, page 101; the time of P falling to C thus determined is $\frac{(S+P)r'^{\frac{3}{2}}}{S^{\frac{3}{2}}} \cdot \frac{\pi}{2\sqrt{2}}$, and the time of S falling to the same point is $\frac{(S+P)(r-r')^{\frac{3}{2}}}{P^{\frac{3}{2}}} \cdot \frac{\pi}{2\sqrt{2}}$. These times are equal, since $S : P :: r-r' : r'$. The velocities may be similarly found.

(51.) *Prop.* LXIII. To determine the motions of the bodies S and P, when projected obliquely, is the object of this Prop. We know from Note (61) the forces acting on P and S, and the point to which they are directed, it remains to determine the velocities and directions of motion of the bodies P and S about C. Let p, s be the original velocities impressed, in directions making angles α, β respectively with the line PS; then these velocities must be considered as the resultants of parallel velocities, v', v'', of the bodies P, S, by which they describe similar ellipses about C, Prop. 57, and of the velocity, c, of the centre of gravity. Let the velocities v', v'' make an angle ϕ with the line PS, and c an angle γ with the same, then we have

$$\left.\begin{array}{ll} p.\cos.\alpha=c.\cos.\gamma+v'.\cos.\phi & s.\cos.\beta=c.\cos\gamma-v''.\cos.\phi \\ p.\sin.\alpha=c.\sin.\gamma+v'.\sin.\phi & s.\sin.\beta=c.\sin.\gamma-v''.\sin.\phi \end{array}\right\} \quad (a)$$

Multiply the two upper of these equations by P, S respectively, and add them, observing that $Pv'=Sv''$, we have

$$(Pp.\cos.\alpha+Ss.\cos.\beta)=(P+S).c.\cos.\gamma \qquad (b)$$

In the same way the two lower equations give us

$$(Pp.\sin.\alpha+Ss.\sin.\beta)=(P+S).c.\sin.\gamma \qquad (c)$$

From equations (*b*) and (*c*), we have

$$c=\frac{\sqrt{P^2p^2+S^2s^2+2.P.S.p.s.\cos.(\alpha-\beta)}}{P+S},$$

$$\tan.\gamma=\frac{Pp.\sin.\alpha+Ss.\sin.\beta}{Pp.\cos.\alpha+Ss.\cos.\beta}.$$

These equations determine for us the velocity and direction of the motion of the centre of gravity.

Again, subtracting the upper of equations (*a*), and also the lower, we have

$$p.\cos.\alpha-s.\cos.\beta=(v'+v'').\cos.\phi=v'.\frac{P+S}{S}.\cos.\phi$$

$$p.\sin.\alpha-s.\sin.\beta=(v'+v'').\sin.\phi=v'.\frac{P+S}{S}.\sin.\phi.$$

Hence we can readily deduce

$$v'=\frac{S}{P+S}.\sqrt{p^2+s^2-2p.s.\cos.(\alpha-\beta)}$$

$$v''=\frac{P}{P+S}.\sqrt{p^2+s^2-2p.s.\cos.(\alpha-\beta)}$$

$$\tan.\phi=\frac{p.\sin.\alpha-S.\sin.\beta}{p.\cos.\alpha-S.\cos.\beta}.$$

Hence we know the velocity and direction of the motion of each of the bodies, P and S, about the centre of gravity, and we can therefore completely construct the orbit it will describe by Prop. 40, page 168, of this Treatise.

NOTES

TO

PART II.

(52.) *Prop.* XXIII. Whenever a body moves subjected to the action of a single centre, the locus of the points fallen from externally is a circle, as will readily appear from Prop. XL. of Newton. This locus sometimes becomes a right line, and sometimes is removed off to an infinite distance; these will, however, readily appear not to be exceptions.

(53.) *Page* 142, *Line* 1. The locus in this case is not always an hyperbola, the expression for the eccentricity is $\frac{4e}{3+e^2}$, which is >1 for all values of e between 1 and 3, $=1$ when $e=3$, and <1 when e is greater than 3. Hence the locus is an hyperbola in the first case, a parabola in the second, and an ellipse in the third.

(54.) *Page* 159, *Art.* 141. The investigation of this locus is erroneous. The velocity of projection is $2\sqrt{gr}.\sin.\theta$, the equation of the parabola is $y=-x.\tan.\theta-m$. With these *data* the locus required is a circle with its centre on the vertical axis, at a distance from the origin $=-(m-\sqrt{gr.t})$, and its radius is $t\sqrt{gr}$.

(55.) *Page* 161. On the subject of motion in a resisting medium, the reader is referred to the second book of the Dynamics of Whewell, where it is most ably and fully treated. Any brief notice of this complicated matter would be insufficient, and a complete discussion unsuitable to the present Treatise.

(56.) *Page* 167, *Subsection* II. In the inverse method of central forces it would be extremely easy to discover an equation between the radius vector and perpendicular on the tangent. Let the force, for instance, be the inverse third power of the distance, we have then from formula (8), page 119, $\frac{h^2 dp}{p^3 dr} = \frac{\mu dr}{r^3}$, whence $\frac{h^2}{p^2} = \frac{\mu}{r^2} + k^2$. This, however, would be but an apparent evasion of the difficulty, as the equation between r and p is implicitly a differential equation. This objection holds fully against the investigation of caustics by reflection, given in Coddington's Optics.

(57.) *Page* 194, *Art.* 209. As an instance of this equation having four possible roots when $n - 1$ is even, let $n = 5$; then the equation will have four possible roots when $\frac{(n-1)^2 . h^4}{4\mu^2} > 4(\epsilon^2 - 1) . a^4$. Only two of these, however, give apsidal distances; and, in fact, the other two are negative, and respectively equal to the two positive ones.

(58.) *Page* 209, *Art.* 229. It is not correct to say, that the orbits can have only two apsides. Strictly speaking, the orbit can have only one apsidal distance in this case, but it may have any number of apsides, as there will possibly be one when $\omega = \frac{2m\pi}{n-3}$, m being any number in the series 0, 1, 2, 3, &c. The curve with all these apsides is only mathematically possible, as, physically speaking, such a curve could not be described with a continuous motion.

We may remark some other properties of this curve. Since $\frac{2}{\Delta'^{n-1}}=\frac{2}{r^{n-1}}$, by reverting to Art. (70.) we find $C=\pm\frac{4r}{n-1}$, which shews that in this class of curves C is always proportional to r. Hence the radius of curvature can be readily found. If the quantity $n-3$ be negative, the equation of the orbit would be of the form $r'^{\frac{n'+3}{2}}.\cos.^{\frac{n'+3}{2}}=a'^{\frac{n'+3}{2}}$; if then $n'+3=n-3$, we shall have $r.r'=a.a'$, which points out a remarkable relation between these two classes of curves.

(59.) The reader would do well to exercise himself in the preceding theory by working questions on it. He will find them in abundance in the volumes that have been published containing the Cambridge and Dublin Problems. Such remarks as the following would be useful:—

1°. If the law of the periodic times in *circles* about any centres be known, the law of force may be found in *any orbit* about the same centre; for let two concentric circles be described cutting the orbit in any two points, the relation of the forces in those circles is known, (Cor. 7, Prop. 4.), but these are the same as the forces in those points of the orbit.

2°. If the law of force, and ratio of the velocity in the curve to that in the equidistant circle be known, the law of velocity may be found in any orbit.

3°. The locus of the points fallen from *externally* to acquire the velocity in the curve, must always be a circle concentric with the centre of force, and consequently this locus must become a right line if the centre of force be infinitely distant. If the radius of the locus be infinite, and the centre of force at a finite distance, the locus of the points fallen to *internally* is a curve of the same species as the given one.

4°. The only curves in which the velocity is equal to that in the equidistant circle, are the equilateral hyperbola and logarithmic spiral.

5°. If the paracentric force be cypher, the curve must be either the equilateral hyperbola or circle.

6°. The paracentric velocity never admits of a *max.* or *min.* when the force is inversely as the third power of the distance.

(60.) On the subject of Attractions the author has restricted himself to mere theory. The application to the figure of the earth and the tides may, with the other parts of Newton which regard the system of the world, afford materials for a second volume of the present work.

(61.) *Page* 211. It is easy to establish the equable description of areas of the earth about the sun, by observing the sun's apparent diameter: this is found to vary in the subduplicate ratio of the apparent angular motion of the sun about the earth, or of the real angular motion of the earth about the sun. Hence the product of the angular motions and squares of the distances is constant. The same mode of establishing Kepler's first law applies to the moon. To establish the second law, let A be the sun's apparent diameter when 90° from greatest or least distance, then observation shews that his apparent diameter at any other point is $A+\delta.\cos.\omega$, ω being the angular distance from perihelion, where his apparent diameter is greatest. But the apparent diameter being inversely as the distance, we have

$$A : A+\delta.\cos.\omega :: r : \tfrac{1}{2}L,$$

whence

$$r=\frac{\frac{1}{2}\mathrm{L.A}}{\mathrm{A}+\delta.\cos.\omega}=\frac{\frac{1}{2}\mathrm{L}}{1+\frac{\delta}{\mathrm{A}}\cdot\cos.\omega};$$

but again, $\mathrm{A}+\delta:\mathrm{A}-\delta::1+e:1-e$, whence

$$\frac{\delta}{\mathrm{A}}=e,$$

therefore, the orbit is an ellipse whose *latus rectum* is L, and eccentricity $\frac{\delta}{\mathrm{A}}$.

The immediate consequences of these laws are, 1°, the velocities in different points of the same orbit are inversely as the perpendiculars on the tangents; 2°, the angular velocities are inversely as the squarės of the distances; 3°, if the orbits be circular, the angular velocities are inversely as the cubes of the distances, &c.

THE END.

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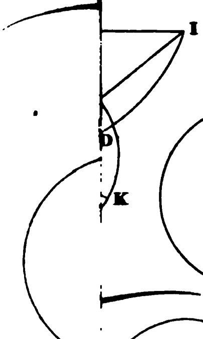

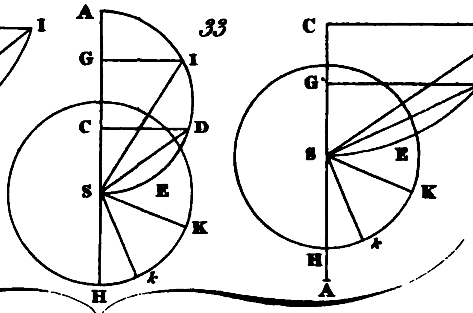
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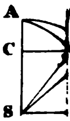
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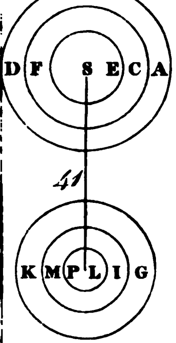
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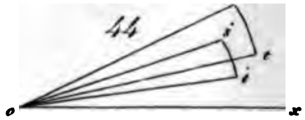
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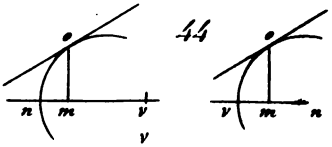
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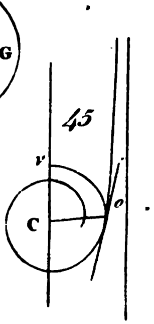
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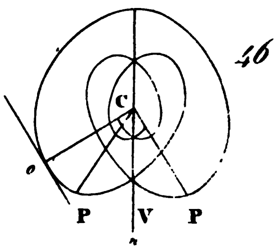
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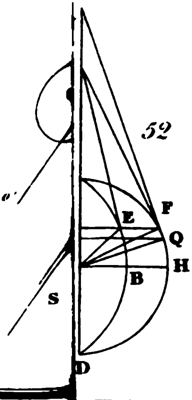
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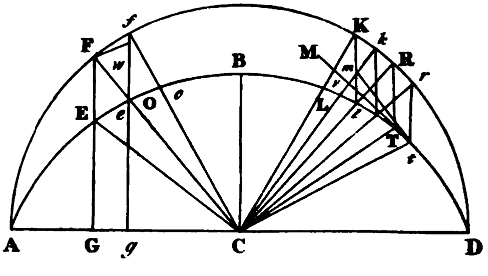
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Walker & Son. Sculpt.

Plate 1

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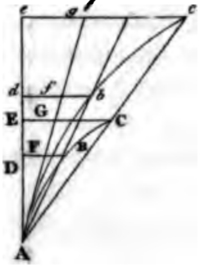

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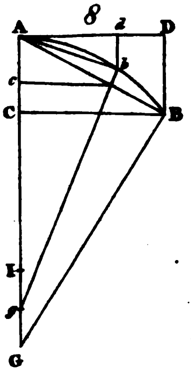

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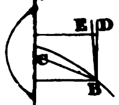

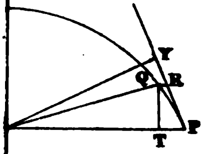

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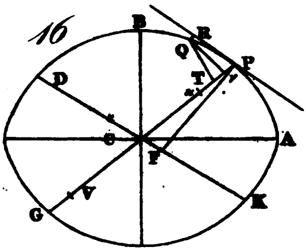

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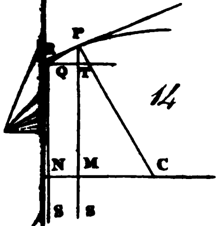

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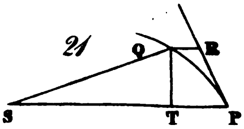

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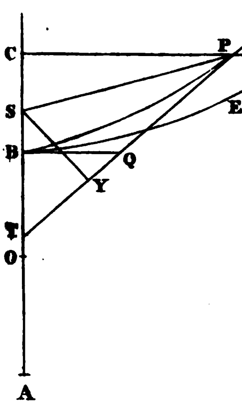

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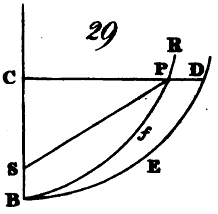

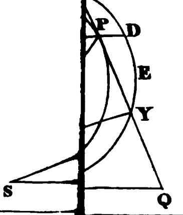

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